Master of Science (Mathematics) (DDE)

Semester – II

Paper Code – 20MAT22C4

PARTIAL DIFFERENTIAL EQUATIONS

DIRECTORATE OF DISTANCE EDUCATION

MAHARSHI DAYANAND UNIVERSITY, ROHTAK

(A State University established under Haryana Act No. XXV of 1975)

NAAC 'A+' Grade Accredited University

Material Production

Content Writer: *Dr. Poonam Redhu*

Maharshi Dayanand University
ROHTAK – 124 001

ISBN :

Price :

Publisher: Maharshi Dayanand University Press

Publication Year : 2021

Paper Code : 20MAT22C4
Partial Differential Equations

M. Marks = 100
Term End Examination = 80
Assignment = 20
Time = 3 Hours

Course Outcomes

Students would be able to:

CO1 Establish a fundamental familiarity with partial differential equations and their applications.
CO2 Distinguish between linear and nonlinear partial differential equations.
CO3 Solve boundary value problems related to Laplace, heat and wave equations by various methods.
CO4 Use Green's function method to solve partial differential equations.
CO5 Find complete integrals of Non-linear first order partial differential equations.

Section-I

Method of separation of variables to solve Boundary Value Problems (B.V.P.) associated with one dimensional Heat equation. Steady state temperature in a rectangular plate, Circular disc, Semi-infinite plate. The Heat equation in semi-infinite and infinite regions. Solution of three dimensional Laplace equations, Heat Equations, Wave Equations in Cartesian, cylindrical and spherical coordinates. Method of separation of variables to solve B.V.P. associated with motion of a vibrating string. Solution of Wave equation for semi-infinite and infinite strings. (Relevant topics from the book by O'Neil)

Section-II

Partial differential equations: Examples of PDE classification. Transport equation – Initial value problem. Non-homogeneous equations. Laplace equation – Fundamental solution, Mean value formula, Properties of harmonic functions, Green function.

Section-III

Heat Equation – Fundamental solution, Mean value formula, Properties of solutions, Energy methods. Wave Equation – Solution by spherical means, Non-homogeneous equations, Energy methods.

Section-IV

Non-linear first order PDE – Complete integrals, Envelopes, Characteristics, Hamilton Jacobi equations (Calculus of variations, Hamilton ODE, Legendre transform, Hopf-Lax formula, Weak solutions, Uniqueness).

Books Recommended:

- I.N. Sneddon, Elements of Partial Differential Equations, McGraw Hill, New York.
- Peter V. O'Neil, Advanced Engineering Mathematics, ITP.
- L.C. Evans, Partial Differential Equations: Second Edition (Graduate Studies in Mathematics) 2nd Edition, American Mathematical Society, 2010.
- H.F. Weinberger, A First Course in Partial Differential Equations, John Wiley & Sons, 1965. M.D. Raisinghania, Advanced Differential equations, S. Chand & Co.

Contents

CHAPTER-0

1 (a) Geometric Notations

(i) $R^n = n-$Dimensional real Euclidean space

(ii) $R^1 = R =$Real line

(iii) $e_i =$ Unit vector in the i^{th} direction$=(0,0,0,...1,...0)$

(iv) A point x in R^n is $x=(x_1,x_2,...,x_n)$

(v) $R^n = \{x=(x_1,x_2,...,x_n)\in R^n \,|x_n>0\}$ =open upper half-space

(vi) A point in R^{n+1} will be denoted as $(x,t)=(x_1,...,x_n,t)$, where t is time variable.

(vii) U,V,W denote open subsets of R^n .We write $V \subset\subset U$ if $V \subset \bar{V} \subset U$ and $\bar{V}$ is compact i.e. V is compactly contained in U.

(viii) ∂U = boundary of U

U=closure of $U = U \cup \partial U$

(ix) $U_T = U \times (0,T]$

(x) $\Gamma_T = \bar{U}_T - U_T$ =parabolic boundary of U_T

(xi) $B^0(x,r)=\{y\in R^n \,\|x-y|<r\}$ = open ball in R^n with centre x and radius r>0

(xii) $B(x,r)=\{y\in R^n \,\|x-y|\le r\}$ =closed ball in R^n with centre x and radius r>0

(xiii) $\alpha(n)$=volume of unit ball $B(0,1)$ in R^n

$$= \frac{r^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$$

$n\alpha(n)$=surface area of unit sphere $B(0,1)$ in R^n

(xiv) If $a,b\in R^n$ s.t. $a=(a_1,a_2,...,a_n)$ and $b=(b_1,b_2,...,b_n)$ then $a,b=\sum_{i=1}^{n} a_i b_i$ and $|a|=\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$

(b) Notations for functions

(i) If $u:U\to R$,we write $u(x)=u(x_1,x_2,...,x_n)$ where $x\in U$, u is smooth if u is infinitely differentiable.

(ii) If u, v are two functions, we write $u\equiv v$ if u, v agree for all arguments

$u \coloneqq v$ means u is equal to v.

(iii) The support of a function u is defined as the set of points where the function is not zero and denoted by spt u.

$$u = \{x \in X \mid f(x) \neq 0\}$$

(iv) The sign function is defined by

$$\operatorname{sgn} x = \begin{cases} 1 & if\ x > 0 \\ 0 & if\ x = 0 \\ -1 & if\ x < 0 \end{cases}$$

$$u^+ = \max(u, 0)$$

$$u^- = -\min(u, 0)$$

$$u = u^+ - u^-$$

$$|u| = u^+ + u^-$$

(v) If $u : U \to R^m$

$u(x) = (u^1(x), \ldots, u^m(x))\ (x \in U)$ where $u = (u^1, u^2, \ldots u^m)$

The function u^i is the ith component of u

(vi) The symbol $\int_{\Sigma} fdS$ denotes the integral of f over $(n-1)$ dimensional surface $\sum$ in R^n

(vii) The symbol $\int_C fdl$ denotes the integral of f over the curve C in R^n

(viii) The symbol $\int_V fdx$ denotes the volume integral of S over $V \in R^n$ and $x \in V$ is an arbitrary point.

(ix) Averages: $\oint_{B(x,r)} fdy = \dfrac{1}{\alpha(n) r^n} \int_{B(x,r)} fdy$

=average of f over ball $B(x,r)$

$$\oint_{\partial B(n,r)} fds = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(n,r)} fds$$

=average of f over surface of ball $B(x,r)$

(x) A function $u : U \to R$ is called Lipschitz continuous if

$|u(x) - u(y)| \leq C|x - y|$, for some constant C and all $x, y \in U$ .We denote

$$Lip[u] = \sup_{\substack{x,y \in U \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|}$$

(xi) The convolution of functions f, g is denoted by $f * g$.

(c) Notations for derivatives: Suppose $u:U\to R, x\in U$

(i) $\dfrac{\partial u(x)}{\partial x_i}=\lim\limits_{h\to 0}\dfrac{u(x+he_i)-u(x)}{h}$

provided that the limit exists. We denote $\dfrac{\partial u}{\partial x}$ by u_{x_i}

Similarly $u_{x_i x_j}=\dfrac{\partial^2 u}{\partial x_i \partial x_j}$ and $u_{x_i x_j x_k}=\dfrac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}$ and in this way higher order derivatives can be defined.

(ii) **Multi-index Notation**

(a) A vector α of the for $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)$ where each α_i is a non-negative integer is called a multi- index of order $|\alpha|=\alpha_1+\alpha_2+...+\alpha_n$

(b) For given multi-index α ,define

$$D^{\alpha}u(x)=\frac{\partial^{\alpha}u(x)}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}}$$

(c) If i is a non-negative integer

$$D^i u(x)=\{D^{\alpha}u(x),|\alpha|=i\}$$

The set of all partial derivatives of order i.

(d) $|D^k u(x)|=\left\{\sum\limits_{|\alpha|=k}|D^{\alpha}u|^2\right\}^{1/2}$

(iii) $\Delta u=\sum\limits_{i=1}^{n}u_{x_i x_i}$

=Laplacian of u

=trace of Hessian Matrix.

(iv) Let $x,y\in R^n$ $i.e. x=(x_1,x_2,...,x_n), y=(y_1,y_2,...,y_n)$

Then we write

$$D_x u=(u_{x_1},...,u_{x_n})$$

$$D_y u=(u_{y_1},...,u_{y_n})$$

The subscript x or y denotes the variable w.r.t. differentiation is being taken

(d) Function Spaces

(i) (a) $C(U)=\{u:U\to R|\text{u is continous}\}$

(b) $C(\bar{U})=\{u\in C(u)|u \text{ is uniformly continous on bounded subsets of } U\}$

(c) $C^k(U)=\{u:U\to R|u \text{ is } k \text{ times continuous differentiable}\}$

(d) $C^k(\overline{U}) = \{u : C^k(U) \mid D^\alpha u$ is uniformly continuous unbounded subsets of U for all $|\alpha| \le k\}$

(e) $C^\infty(U) = \left\{u : U \to R \middle| u \quad is \quad \inf initly \quad differentiable\right\}$

(ii) $C_c(U)$ means $C(U)$ has compact support.

Similarly, $C_c^k(U)$ means $C^k(U)$ has compact support.

(iii) The function $u : U \to R$ is Lebsegue measurable over L^p if $\|u\|_{L^p(U)} < \infty$

$$\|u\|_{L^p(U)} = \left(\int_U |u|^p \, dx\right)^{1/p}, 1 \le p \le \infty$$

The function $u : U \to R$ is Lebsegue measurable over L^∞ if $\|u\|_{L^\infty(U)} < \infty$

$$\|u\|_{L^\infty(U)} = ess \sup_U |u|$$

(iv) $L^p(U) = \left\{u : U \to R \middle| u \quad is \quad Lebsegue \quad measurable\, over \quad L^p\right\}$

$L^\infty(U) = \left\{u : U \to R \middle| u \quad is \quad Lebsegue \quad measurable\, over \quad L^\infty\right\}$

(v) $\|Du\|_{L^p(U)} = \big\| |Du| \big\|_{L^p(U)}$

Similarly, $\left\|D^2 u\right\|_{L^p(U)} = \left\| \left|D^2 u\right| \right\|_{L^p(U)}$

(vi) If $u : U \to R^m$ is a vector, where $u = \left(u^1, u^2, \ldots, u^m\right)$ then $D^k u = \left\{D^\alpha u, |\alpha| = k\right\}$

similarly other operator follow.

(e) Notation for estimates:

(i) Big Oh(O)order

We say

$f = O(g)$ as $x \to x_0$ provided there exists a constant C such that $|f(x)| \le C|g(x)|$, for all x sufficiently close to x_0.

(ii) Little Oh(o) order

We say

$f = o(g)$ as $x \to x_0$, provided $\underset{x \to x_0}{lt} \left|\dfrac{f(x)}{g(x)}\right| \to 0$

2 Inequalities

(i) Convex Function

A function $f : R^n \to R$ is said to convex function if

$$f(\tau x + (1-\tau)y) \le \tau f(x) + (1-\tau) f(y)$$

for all $x, y \in R^n$ and each $0 < \tau < 1$.

(ii) Cauchy's Inequality

$$ab \le \frac{a^2}{2} + \frac{b^2}{2} \qquad (a, b \in R)$$

(iii) Holder's Inequality

Let $1 \le p, q \le \infty$; $\frac{1}{p} + \frac{1}{q} = 1$, $\qquad u \in L^p(u), v \in L^q(u)$

$$\int_U |uv|\, dx \le \|u\|_{L^p(U)} \|v\|_{L^q(U)}$$

(iv) Minkowski's Inequality

Let $1 \le p \le \infty$, and $u, v \in L^p(U)$, $\quad$ Then $\|u+v\|_{L^p(U)} \le \|u\|_{L^p(U)} + \|v\|_{L^q(U)}$

(v) Cauchy Schwartz Inequality

$$|x.y| < |x||y| \qquad (x, y \in R^n)$$

3 Calculus

(a) Boundaries

Let $U \subset R^n$ be open and bounded, k={1,2,…,}

Definitions:

(i) The boundary ∂U is C^k if for each point $x^0 \in \partial U$ there exists r>0 and a C^k function $\Upsilon : R^{n-1} \to R$ such that $U \cap B(x^0, r) = \{x \in B(x^0, r) | x_n > \Upsilon(x_1, ..., x_{n-1})\}$

Also, ∂U is analytic if Υ is analytic.

(ii) If U is C^1, then along ∂U , the outward unit normal at any point $x_0 \in \partial U$ is denoted by $\underline{v}(x^0) = (v_1, ..., v_n)$.

(iii) Let $u \in C^1(\bar{U})$ then normal derivative of u is denoted by $\frac{\partial u}{\partial v} = \underline{v}.Du$

(b) Gauss-Green Theorem

Let U be a bounded open subset of R^n and ∂U is C^1 . $u : U \to R^n$ and also $u \in C^1(\bar{U})$ then

$$\int_U u_{x_i}\, dx = \int_{\partial U} u v^i\, dS \qquad (i = 1, 2, ..., n)$$

(c) Integration by parts formula

Let $u, v \in C^1\left(\bar{U}\right)$ then

$$\int_U u_{x_i} v dx = -\int_U u v_{x_i} dx + \int_{\partial U} u v \nu^i dS$$

Proof: By Gauss-Green's Theorem

$$\int_U (uv)_{x_i} dx = \int_{\partial U} (uv) \nu^i dS$$

Or $$\int_U u_{x_i} v dx + \int_U u v_{x_i} dx = \int_{\partial U} (uv) \nu^i dS$$

Or $$\int_U u_{x_i} v dx = -\int_U u v_{x_i} dx + \int_{\partial U} (uv) \nu^i dS$$

(d) Green's formula

Let $u, v \in C^2\left(\bar{U}\right)$ then

(i) $\displaystyle\int_U \Delta u dx = \int_{\partial U} \frac{\partial u}{\partial \nu} dS$

Proof: $\displaystyle\int_U \Delta u dx = \int \left(u_{x_i}\right)_{x_i} dx$

Integrating by parts, taking the second function as unity

$$\int_U \Delta u dx = \int_{\partial U} u_{x_i} \nu^i dS$$

$$= \int_{\partial U} \frac{\partial u}{\partial \nu} dS$$

Hence proved.

(ii) $\displaystyle\int_U Du.Dv dx = -\int_U u \Delta v dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u dS$

Proof: $\displaystyle\int_U Du.Dv dx = -\int_U u \Delta v dx + \int_{\partial U} u Dv.\nu dS$

$$= -\int_U u \Delta v dx + \int_{\partial u} u \frac{\partial v}{\partial \nu} dS \qquad \text{(integrating by parts)}$$

(iii) $\displaystyle\int_U \left(u \Delta v - v \Delta u\right) dx = \int_{\partial U} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu}\right) dS$

Proof: $\displaystyle\int_U u \Delta v dx = -\int_U Du.Dv dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u dS$

Similarly, $\displaystyle\int_U v \Delta u dx = -\int_U Du.Dv dx + \int_{\partial U} \frac{\partial u}{\partial \nu} u dS$

subtracting, we get the result.

(e) Conversion of n-dimensional integrals into integral over sphere

(i) Coarea formula

Let $u : R^n \to R$ be Lipschitz continuous and assume that for a.e. $r \in R$,the level set $\{x \in R^n \,|u(x) = r\}$ is a smooth and n-1 dimensional surface in R^n .Suppose also $f : R^n \to R$ is smooth and summable. Then

$$\int_{R^n} f\,|Du|\,dx = \int_{-\infty}^{\infty}\left(\int_{\{u=r\}} fdS\right)dr$$

Cor. Taking $u(x) = |x - x_0|$

Let $f : R^n \to R$ be continuous and summable then

$$\int_{R^n} fdx = \int_0^{\infty}\left(\int_{\partial B(x_0,r)} fdS\right)dr$$

for each point $x_0 \in R^n$ or we can say

$$\frac{d}{dr}\left(\int_{B(x_0,r)} fdx\right) = \int_{\partial B(x_0,r)} fdS$$

for each r>0.

(f) To construct smooth approximations to given functions

Def: If $U \subset R^n$ is open, given $\varepsilon > 0$ .We define $U_\varepsilon := \{x \in U \,|dist(x, \partial U) > \varepsilon\}$

Def. Standard Mollifier

Let $\eta \in C^\infty(R^n)$ such that

$$\eta(x) := \begin{cases} c\exp\left(\dfrac{1}{|x|^2 - 1}\right) if & |x| < 1 \\ 0\, if & |x| > 1 \end{cases}$$

The constant c is chosen so that $\int_{R^n} \eta dx = 1$

Def. We define

$\eta_\varepsilon(x) := \dfrac{1}{\varepsilon^n}\eta\left(\dfrac{x}{\varepsilon}\right)$ for every $\varepsilon > 0$.

Properties:

(i) The functions η_ε are C^∞ since $\eta(x)$ are C^∞ .

(ii) $\int_{R^n} \eta_\varepsilon dx = \dfrac{1}{\varepsilon^n}\int_{R^n} \eta\left(\dfrac{x}{\varepsilon}\right)dx$

$= \int_{R^n} \eta(x)\,dx$ (by definition of n-tuple integral)

$=1$

(g) Mollification of a function

If $f:U \to R$ is locally integrable

We define the mollification of f

$f^{\varepsilon} := \eta_{\varepsilon} * f$ in U_{ε}

$$= \int_U \eta_{\varepsilon}(x-y)\, f(y)\, dy \quad = \int_{B(0,\varepsilon)} \eta_{\varepsilon}(y)\, f(x-y)\, dy \qquad \text{(by definition)}$$

Properties:

(i) $f^{\varepsilon} \in C^{\infty}(U_{\varepsilon})$

(ii) $f^{\varepsilon} \to f$ almost everywhere as $\varepsilon \to 0$

(iii) If $f \in C(U)$ then $f^{\varepsilon} \to f$ uniformly on compact subset of U almost everywhere.

Function Analysis Concepts

(i) L^p **space**: Assume U to be a open subset of R^n and $1 \le p \le \infty$ .If $f:U \to R$ is measurable, we define

$$\|f\|_{L^p(U)} := \begin{cases} \left(\int_U |f|^p\, dx \right)^{1/p} & if \quad 1 \le p < \infty \\ ess \sup_U |f| & if \quad p = \infty \end{cases}$$

Transformation from Ball $B(x,r)$ **to unit Ball** $B(0,1)$

Let $B(x,r)$ be a ball with centre x and radius r and $B(0,1)$ be an arbitrary point of $B(x,r)$ and z be an arbitrary point of $B(0,1)$ then relation between y and z is y=x+rz.

CHAPTER-1

HEAT, WAVE AND LAPLACE EQUATIONS

Structure

1.1 Introduction

1.2 Method of separation of variables to solve B.V.P. associated with one-dimensional Heat equation

1.3 Steady state temperature in a rectangular plate, Circular disc and semi-infinite plate

1.4 Solution of Heat equation in semi-infinite and infinite regions

1.5 Solution of three dimensional Laplace, Heat and Wave equations in Cartesian, Cylindrical and Spherical coordinates.

1.6 Method of separation of variables to solve B.V.P. associated with motion of a vibrating string

1.7 Solution of wave equation for semi-infinite and infinite strings

1.1 Introduction

In this section, the temperature distribution is studied in several cases. For finding the temperature distribution we require to solve the Heat equation with different Boundary Value Problem (B.V.P.), whereas to find the steady state temperature distribution we require to attempt a solution of Laplace equation and to obtain motion of vibrating string we find a solution of Wave equation.

1.1.1 Objective

The objective of these content is to provide some important results to the reader like:

(i) Temperature distribution in a bar with ends at zero temperature, insulated ends, radiating ends and ends at different temperature.
(ii) Steady state Temperature distribution in a finite, semi-infinite and infinite plate
(iii) Heat conduction in semi-infinite and infinite bar
(iv) Solution of Heat, Laplace and Wave equation in various cases

1.2. Method of Separation of Variables to solve B.V.P. associated with One Dimensional Heat Equation

A parabolic equation of the type

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \qquad \text{---(1)}$$

k being a dissasivity (constant) and $u(x,t)$ being temperature at a point (x,t) of a solid at time t is known as Heat Equation in one dimension.

We now proceed to discuss the method of separation of variables to solve B.V.P., with boundary conditions:

$$u(0,t)=0 \text{ and } u(l,t)=0 \qquad ...(2)$$

and

$$u(x,0)=f(x) \text{ and } \left[\frac{\partial u}{\partial t}\right]_{t=0}=\nu(x) \qquad ...(3)$$

Suppose the solution of (1) is

$$u(x,t)=X(x)T(t) \qquad ...(4)$$

where X(x) is a function of x only and T(t) is a function of t only.
Therefore, we have

$$\frac{\partial u}{\partial x}=\frac{dX}{dx}T(t)$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{d^2X}{dx^2}T(t) \qquad ...(5)$$

and

$$\frac{\partial u}{\partial t}=X(x)\frac{dT}{dt}$$

Inserting (5) into (1), we obtain

$$T(t)\frac{d^2X}{dx^2}=\frac{1}{k}X(x)\frac{dT}{dt}$$

Dividing both sides by u(x,t)=X(x)T(t),we have

$$\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{kT}\frac{dT}{dt} \qquad ...(6)$$

Now, L.H.S. of (6) is independent of t and R.H.S. is independent of x, either side of (6) can be equated to some constant of separation. If constant of separation is p^2, then

$$\frac{1}{X}\frac{d^2X}{dx^2}=p^2 \quad \text{and} \quad \frac{1}{kT}\frac{dT}{dt}=p^2$$

$$\text{or } \frac{d^2X}{dx^2}-p^2X=0 \qquad ...(7)$$

$$and\ \frac{dT}{dt}-p^2kT=0 \qquad ...(8)$$

These equations have the solutions

$$X(x)=c_1e^{px}+c_2e^{-px} \text{ and } \mathrm{T}(t)=Ae^{kp^2t} \qquad ...(9)$$

In view of (2), (4) implies

$$u(0,t)=0 \ \Rightarrow \mathrm{X}(0)T(t)=0$$

Here, either X(0)=0 or T(t)=0. If T(t) is assumed to be zero identically then u(x,t)=X(x)T(t) is zero identically, that is the temperature function is zero identically, which is of no interest. Thus, we take

X(0)=0

Similarly, $u(l,t)=0 \Rightarrow \mathrm{X}(l)T(t)=0 \Rightarrow \mathrm{X}(l)=0$

Thus, we have

$$\mathrm{X}(0)=\mathrm{X}(l)=0 \qquad ...(10)$$

Now, applying (10) on (9), we get

$$c_1+c_2=0 \text{ and } c_1e^{pl}+c_2e^{-pl}=0$$

This system has a trivial solution

$$c_1=c_2=0$$

and so X(x)=0, then the temperature function becomes zero which is not being assumed.

Now, let $p^2=0,$ then (7) and (8) implies

$$\frac{d^2X}{dx^2}=0 \text{ and } \frac{dT}{dt}=0$$

$$\Rightarrow X(x)=c_1x+c_2 \text{ and } \mathrm{T}(t)=c \qquad ...(11)$$

Now, applying (10) on (11), we obtain:

$$c_1=c_2=0$$

$$\Rightarrow X(x)=0$$

Again, the temperature function becomes zero and is of no interest.

So, assume that the constant of separation is - p^2, so that

$$\frac{d^2X}{dx^2}+p^2X=0 \qquad ...(12)$$

$$\frac{dT}{dt}+kp^2T=0 \qquad ...(13)$$

Solution of (12) is

$$X(x)=c_1\cos px+c_2\sin px \qquad ...(14)$$

In view of (10), (14) implies

$$X(0)=c_1=0 \text{ and}$$

$$\mathrm{X}(l)=c_2\sin pl=0$$

$$\Rightarrow pl=n\pi \text{ for n} \neq 0\text{, n being an integer.}$$

$$\Rightarrow p=\frac{n\pi}{l}$$

For n≠0, we have infinite many solutions

$$X_n(x) = a_n \sin\frac{n\pi x}{l} \text{ ; n =1,2,. ..} \qquad ...(15)$$

Now, for $p = \frac{n\pi}{l}, (13)$ gives

$$\frac{dT}{dt} + k\left(\frac{n\pi}{l}\right)^2 T = 0$$

$$or \;\; \frac{dT}{dt} + \lambda_n T = 0, \text{ where } \lambda_n = \frac{kn^2\pi^2}{l^2}$$

Its general solution is

$$T(t) = c_n e^{-\lambda_n t} \qquad ...(16)$$

Combining (15) and (16), we have

$$u_n(x,t) = c_n e^{-\lambda_n t} a_n \sin\frac{n\pi x}{l} \qquad ...(17)$$

where n =1,2,…

Now, for the general solution, we have

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n t} \sin\frac{n\pi x}{l} \qquad ...(18)$$

$$where \;\; \mathrm{b}_n = a_n c_n$$

giving

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = f(x) \qquad ...(19)$$

$$and \left[\frac{\partial u}{\partial t}\right]_{t=0} = \left[\sum_{n=1}^{\infty}\left\{-\lambda_n b_n \sin\frac{n\pi x}{l}.e^{-\lambda_n t}\right\}\right]_{t=0}$$

$$= -\sum_{n=1}^{\infty} \lambda_n b_n \sin\frac{n\pi x}{l} = v(x) \qquad ...(20)$$

From (19) and (20), the constant b_n can be determined easily and thus, (18) represents the solution of Heat equation.

1.2.1 Ends of the Bar Kept at Temperature Zero

Suppose we want the temperature distribution u(x,t) in a thin, homogeneous bar of length L, given that the initial temperature in the bar at time zero in the section at perpendicular to the x-axis is specified by u(x,0)=f(x). The ends of the bar are maintained at temperature zero for all time. The boundary value problem modeling this temperature distribution is

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}\quad (0<x<L,t>0) \qquad ...(1)$$
$$u(0,t)=u(L,t)=0\quad (t>0) \qquad ...(2)$$
$$u(x,0)=f(x)\quad (0<x<L) \qquad ...(3)$$

Put $u(x,t)=X(x)T(t) \qquad ...(4)$

into the equation (1) to get

$$XT'=a^2X''T \qquad ...(5)$$

where primes denote differentiation w.r.t. the variable of the function.

Then, $\dfrac{X''(x)}{X(x)}=\dfrac{T'(t)}{a^2T(t)} \qquad ...(6)$

The R.H.S. of this equation is a function of t only and L.H.S. a function of x only and these variables are independent. We could, e.g. choose any t, we like, thereby fixing the right side of the equation at a constant value. The left side would then have to equal this constant for all x. We therefore, conclude that $\frac{X''}{X}$ is constant. But then $\frac{T'}{a^2T}$ must equal the same constant, which we will designate $-\lambda$ (The negative sign is a convention; we would eventually get the same solution if we used λ). The constant λ is called the separation constant.

Thus, we have

$$\frac{X''}{X}=\frac{T'}{a^2T}=-\lambda$$

giving us two ordinary differential equations

$$X''+\lambda X=0$$
$$T'+\lambda a^2T=0$$

Now consider the boundary conditions. First

$$u(0,t)=X(0)T(t)=0$$
$$\Rightarrow X(0)=0 \text{ or } \mathrm{T}(t)=0$$

If $\mathrm{T}(t)=0$ for all t, then the temperature in the bar is always zero. This is indeed the solution if f(x)=0. Otherwise, we must assume that T(t) is non-zero for some t and conclude that

$$X(0)=0$$

Similarly,
$$u(L,t)=X(L)T(t)=0$$
$$\Rightarrow X(L)=0$$

We now have the following problems for X and T

$$X''+\lambda X=0$$
$$X(0)=X(L)=0$$
and $$T'+\lambda a^2 T=0$$

We will solve for $X(x)$ first because we have the most information about X. The problem is a regular Strum-Liouville Problem on [0,L]. A value for λ for which the problem has a non-trivial solution is called an eigen value of this problem. For such a λ , any non-trivial solution for X is called an eigen function.

Case 1: $\lambda=0$

Then, $X''=0$, so $X(x)=cx+d$, Now $X(0)=d=0$, so $X(x)=cx$. But then $X(L)=cL=0 \Rightarrow c=0$

Thus, there is only the trivial solution for this case. We conclude that 0 is not an eigen value of problem.

Case 2: $\lambda<0$

Write $\lambda=-k^2$, with k >0. Then, equation for X(x) is

$$X''-k^2X=0$$

with general solution

$$X(x)=ce^{kx}+de^{-kx}$$

Now, $X(0)=c+d=0 \Rightarrow c=-d$

Therefore, $X(x)=ce^{kx}-ce^{-kx}=c\left(e^{kx}-e^{-kx}\right)$

Next, $X(L)=c\left(e^{kL}-e^{-kL}\right)=0$

Here, $e^{kL}-e^{-kL}\neq 0$, because kL>0, so c=0. Therefore, there are no nontrivial solutions of the problems if $\lambda<0$, and this problem has no negative eigen value.

Case 3: $\lambda>0$

Write $\lambda=k^2$, with k>0. The general solution of

$$X''+k^2X=0$$

is $X(x)=c\cos kx+d\sin kx$

Now, $X(0)=c=0$, so $X(x)=d\sin kx$.

Therefore, $X(L)=d\sin kL=0$

To have a non-trivial solution, we must be able to choose $d\neq 0$.

This require that $\sin kL = 0$, which occurs if kL is a positive integer multiple of π,

say $kL = n\pi$.

Thus, choose $k = \frac{n\pi}{L}$, for n=1,2,…

For each such n, we can choose

$$X_n(x) = d_n \sin\left(\frac{n\pi x}{L}\right)$$

This is a eigen function of the given problem corresponding to the eigen value $\lambda = k^2 = \frac{n^2\pi^2}{L^2}$

Now, return to the problem for T with $\lambda = \frac{n^2\pi^2}{L^2}$, the differential equation is

$$T' + \frac{n^2\pi^2 a^2 T}{L^2} = 0$$

with general solution

$$T_n(t) = a_n e^{\frac{-n^2\pi^2 a^2 t}{L^2}}$$

For each positive integer n, we can get

$$u_n(x,t) = c_n \sin\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2 a^2 t}{L^2}}, \text{ where } c_n = a_n d_n$$

This function satisfies the heat equation and the boundary conditions $u(0,t) = u(L,t) = 0$ on $t \geq 0$ To satisfy the initial condition for a given n, however, we need

$$u_n(x,0) = c_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

And this is possible only if $f(x)$ is a constant multiple of this sine function. Usually, to satisfy the initial condition we must attempt a superposition of all the u_n's:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2 a^2 t}{L^2}}$$

The initial condition now requires that

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

Which we recognize as the Fourier sine expansion of $f(x)$ on [0.L]. Therefore, choose the c_n's as the Fourier sine coefficients of $f(x)$ on [0,L]:

$$c_n = \frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi x}{L}\right)d\xi$$

With certain conditions on $f(x)$ this Fourier sine series converges to $f(x)$ for $0 < x < L$ and the formal solution of the boundary value problem is

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right)\sin\left(\frac{n\pi x}{L}\right)e^{\frac{-n^2\pi^2a^2t}{L^2}}$$

Example: As a special example, suppose the bar is kept at constant temperature A, except at its ends, which are kept at temperature zero. Then,

$$f(x) = A \qquad (0 < x < L)$$

and

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L A\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2A}{n\pi}(1-\cos n\pi) \\ &= \frac{2A}{n\pi}\left(1-(-1)^n\right) \end{aligned}$$

The solution in this case is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty}\frac{2A}{n\pi}\left[1-(-1)^n\right]\sin\left(\frac{n\pi x}{L}\right)e^{\frac{-n^2\pi^2a^2t}{L^2}} \\ &= \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi x}{L}\right)e^{\frac{-(2n-1)^2\pi^2a^2t}{L^2}} \end{aligned}$$

We got the last summation from the preceding line by noticing that $1-(-1)^n = 0$ if n is even, so all the terms in the series vanish for n even and we need only retain the terms with n odd. This is done by replacing n with $2n-1$, there by summing over only the odd positive integers.

Problems: Solve the following boundary value problem:

1. $\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L,\ t > 0)$

$u(0,t) = u(L,t) = 0 \qquad (t > 0)$

$u(x,0) = x(L-x) \qquad (0 < x < L)$

2. $$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L, t > 0)$$
$$u(0,t) = u(L,t) = 0 \qquad (t > 0)$$
$$u(x,0) = x^2(L - x) \qquad (0 < x < L)$$

3. $$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L, t > 0)$$
$$u(0,t) = u(L,t) = 0 \qquad (t > 0)$$
$$u(x,0) = L\left[1 - \cos\left(\frac{2\pi x}{L}\right)\right] \qquad (0 < x < L)$$

1.2.2 Temperature in a Bar with Insulated Ends

Consider heat conduction in a bar with insulated ends, hence no energy loss across the ends. If the initial temperature is given by $f(x)$, then the temperature function is modeled by the B.V.P.

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L, t > 0)$$
$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \qquad (t > 0)$$
$$u(x,0) = f(x) \qquad (0 < x < L)$$

We will solve for $u(x,t)$, leaving out some details, which are the same as in the preceding problem. Set

$$u(x,t) = X(x)T(t)$$

And substitute into the heat equation to get

$$\frac{X''}{X} = \frac{T'}{a^2 T} = -\lambda$$

In which λ is the separation constant. Then,

$$X'' + \lambda X = 0$$

and $T' + \lambda a^2 T = 0$

as before. Also,

$$\frac{\partial u}{\partial x}(0,t) = X'(0)T(t) = 0$$

implies that $X'(0) = 0$. The other boundary condition implies that $X'(L) = 0$. The other boundary condition implies that $X'(L) = 0$. The problem for X is therefore

$$X''+\lambda X=0 \qquad ...(1)$$
$$X'(0)=X'(L)=0 \qquad ...(2)$$

We seek values of λ for which this problem has non-trivial solutions.

Consider cases on λ:

Case 1: $\lambda=0$

The general solution for (1) is

$$X(x)=cx+d$$

Since $X'(0)=0=c$, therefore, 0 is an eigen value of (1) with eigen function.

$$X(x)=\text{constant}\neq 0$$

Case 2: $\lambda<0$

Write $\lambda=-k^2$ with $k>0$. Then, $X''-k^2X=0$, with general solution

$$X(x)=ce^{kx}+de^{-kx}$$

Now,

$$X'(0)=kc-kd=0\Rightarrow c=d \quad [\because k>0]$$
$$\therefore X(x)=c\left(e^{kx}+e^{-kx}\right)$$

Next,

$$X'(L)=ck\left(e^{kL}-e^{-kL}\right)=0$$

This is zero only if c=0. But this forces $X(x)=0$, so choosing λ negative eigen value.

Case 3: $\lambda>0$

Set $\lambda=k^2$, with $k>0$ .Then,

$$X''+k^2X=0$$

with general solution

$$X(x)=c\cos kx+d\sin kx$$

Now, $X'(0)=dk=0$

implies that d=0. Then, $X(x)=c\cos kx$.

Next, $X'(L)=-ck\sin kL=0$

In order to get a non-trivial solution, we need $c \neq 0$, and must choose k so that $\sin kL = 0$, therefore $kL = n\pi$

for n, a positive integer, and this problem has eigen values

$$\lambda = k^2 = \frac{n^2\pi^2}{L^2}\text{; for n=1,2,…}$$

Corresponding to such an eigen value, the eigen function is

$$X_n(x) = c_n \cos\left(\frac{n\pi x}{L}\right)\text{, for n=1,2,…}$$

We can combine case 1 and case 3, by writing the eigen values as

$$\lambda = \frac{n^2\pi^2}{L^2} \quad \text{for n=0,1,2,...}$$

and eigen functions as

$$X_n(x) = c_n \cos\left(\frac{n\pi x}{L}\right)$$

This is a constant functions, corresponding to $\lambda = 0$, when n=0.

The equation for T is

$$T' + \frac{n^2\pi^2a^2T}{L^2} = 0$$

When n=0, this has solutions

$$T_0(t) = \text{constant} = d_0$$

If n=1,2,…, then

$$T_n(t) = d_n e^{\frac{-n^2\pi^2a^2T}{L^2}}$$

Now let

$$u_0(x,t) = \text{constant} = a_0$$

and $u_n(x,t) = a_n \cos\left(\frac{n\pi x}{L}\right)e^{\frac{-n^2\pi^2a^2t}{L^2}}$, where $a_n = c_n d_n$

Each of these functions satisfies the heat equation and boundary conditions. To satisfy the initial condition, we must usually attempt a superposition of these functions:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)e^{\frac{-n^2\pi^2a^2t}{L^2}}$$

We must choose the a_n's so that

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = f(x)$$

This is a Fourier cosine expansion of $f(x)$ on [0,L], so choose

$$a_0 = \frac{1}{L}\int_0^L f(\xi)\,d\xi$$

and, for n=1,2,…

$$a_n = \frac{2}{L}\int_0^L f(\xi)\cos\left(\frac{n\pi\xi}{L}\right)d\xi$$

The solution is

$$u(x,t) = \frac{1}{L}\int_0^L f(\xi)\,d\xi + \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(\xi)\cos\left(\frac{n\pi\xi}{L}\right)d\xi\right)\cos\left(\frac{n\pi x}{L}\right)e^{\frac{-n^2\pi^2a^2t}{L^2}}$$

Example: Suppose the left half of the bar is initially at temperature A and the right half at temperature zero. Then,

$$f(x) = \begin{cases} A & , 0<x<\dfrac{L}{2} \\ 0 & , \dfrac{L}{2} < x < L \end{cases}$$

$$\therefore \quad a_0 = \frac{1}{L}\int_0^{\frac{L}{2}} A\,d\xi = \frac{A}{2}$$

$$\text{and } \; a_n = \frac{2}{L}\int_0^{\frac{L}{2}} A\cos\left(\frac{n\pi\xi}{L}\right)d\xi = \frac{2A}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

The solution for this temperature distribution is

$$u(x,t) = \frac{A}{2} + \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{L}\right)e^{\frac{-n^2\pi^2a^2t}{L^2}}$$

Since $\sin\left(\frac{n\pi}{2}\right)$ is zero if n is even and equals $(-1)^{k+1}$ if n=2k+1. We may omit all terms of this series in which the summation index is even, and sum over only the odd positive integers. This is done by replacing n with 2n-1 in the function being summed. Then,

$$u(x,t) = \frac{A}{2} + \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\cos\left(\frac{(2n-1)\pi x}{L}\right)e^{\frac{-(2n-1)^2\pi^2a^2t}{L^2}}$$

Problems:

Solve the following B.V.P.'s:

1. $$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \qquad (0<x<\pi, t>0)$$
$$\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(\pi,t)=0 \qquad (t>0)$$
$$\mathrm{u}(x,0)=\sin x \qquad (0<x<\pi)$$

2. $$\frac{\partial u}{\partial t}=4\frac{\partial^2 u}{\partial x^2} \qquad (0<x<2\pi, t>0)$$
$$\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(2\pi,t)=0 \qquad (t>0)$$
$$\mathrm{u}(x,0)=x(2\pi-x) \qquad (0<x<2\pi)$$

3. A thin homogeneous bar of length L has insulated ends initial temperature B, a positive constant. Find the temperature distribution in the bar.

4. A thin homogeneous bar of length L has initial temperature equal to a constant B and the right end (x=L) is insulated, while the left end is kept at a zero temperature. Find the temperature distribution in the bar.

5. A thin homogeneous bar of thermal diffusivity 9 and length 2 cm and insulated has its left end maintained at temperature zero, while the right end is perfectly insulated. The bar has an initial temperature given by $f(x)=x^2$ for 0<x<2. Determine the temperature distribution in the bar. What is $\lim\limits_{t\to\infty} u(x,t)$?

1.2.3 Temperature Distribution in a Bar with Radiating End

Consider a thin, homogeneous bar of length *L*, with the left end maintained at temperature zero, while the right end radiates energy into the surrounding medium, which also is kept at temperature zero. If the initial temperature in the bar's cross section at *x* is *f(x)*, then the temperature distribution is modeled by the B.V.P.

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2} \qquad (0\le x\le L, t>0)$$
$$u(0,t)=0, \quad \frac{\partial u}{\partial x}(L,t)=-Au(L,t) \qquad (t\ge 0)$$
$$u(x,0)=f(x) \qquad (0\le x\le L)$$

The boundary condition at L assumes that heat energy radiates from this end at a rate proportional to the temperature at that end of the bar, A is a positive constant called the transfer co-efficient.

Let $u(x,t)=X(x)T(t)$ to obtain, as before,

$$X''+\lambda X=0$$
$$T'+\lambda a^2 T=0$$

Since,

$$u(0,t) = X(0)T(t) = 0, \text{ then}$$
$$X(0) = 0$$

as $T(t) = 0$, implies that $u(x,t) = 0$ which is possible only if $f(x) = 0$. The condition at the right end of the bar implies that

$$X'(L)T(t) = -AX(L)T(t)$$
$$\Rightarrow X'(L) + AX(L) = 0$$

The problem for $X(x)$ is therefore,

$$X'' + \lambda X = 0$$
$$X(0) = X'(L) + AX(L) = 0$$

From the strum-Liouville theorem, we can be confident that this problem has infinitely many eigen values $\lambda_1, \lambda_2, \ldots,$ each of which is associated with a non-trivial solution, or eigen functions, $X_n(x)$. We would like, however, to know these solutions, so we will consider cases:

Cases 1: $\lambda = 0$,

Then, the solution for $X(x)$ is

$$X(x) = cx + d$$

Since, $X(0) = 0 = d$, then

$$X(x) = cx$$

But then

$$X'(x) = c = -AX(L) = -AcL$$

Then,

$$c(1 + AL) = 0$$

But $1 + AL > 0$, so c=0 and we get only the trivial solution from this case. This means that 0 is not an eigen value of this problem.

Case 2: $\lambda < 0$, write $\lambda = -k^2$, with $k > 0$ .Then,

$$X'' - k^2 X = 0, \text{ so}$$

$$X(x) = ce^{kx} + de^{-kx}$$

Now, $X(0) = c + d = 0 \Rightarrow d = -c.$

$$\therefore X(x) = c\left(e^{kx} - e^{-kx}\right) = 2c\sinh(kx)$$

Then, $X'(L) = 2ck\cosh(kL) = -Ac\sinh(kL)$

To have a non-trivial solution, we must have $c \neq 0$ and this requires that

$$2k\cosh(kL) + A\sinh(kL) = 0$$

This is impossible because $Lk > 0,$ so the left side of this equation is a sum of positive numbers. Therefore, this problem has no negative eigen value.

Case 3: $\lambda > 0,$ write $\lambda = k^2,$ with $k > 0$. Then,

$$X'' + k^2 X = 0, \text{ so}$$

$$X(x) = c\cos kx + d\sin kx$$

Now, $X(0) = c = 0,$ so $X(x) = d\sin kx.$

Further, $X'(L) + AX(L) = dk\cos(kL) + Ad\sin(kL) = 0$

To have a non-trivial solution, we must have $d \neq 0$, and this requires that

$$k\cos(kL) + A\sin(kL) = 0$$

or $\tan(kL) = \dfrac{-k}{A}$

Let $z = kL$. Then, this equation is

$$\tan(z) = \frac{-z}{AL}$$

Since $k = \dfrac{z}{L},$ then $\lambda_n = \dfrac{z_n^2}{L^2}$

is an eigen value of this problem for each positive integer n which is shown in Figure below,

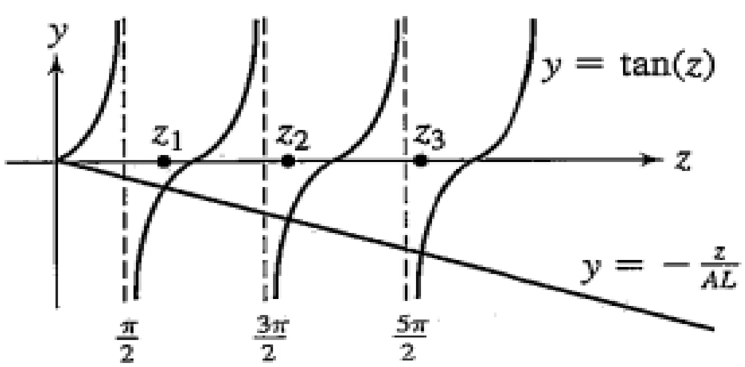


Figure: The eigen values of the problem for a bar with radiating ends with corresponding eigen function

$$X_n(x) = a_n \sin\left(\frac{z_n x}{L}\right)$$

The equation for T is

$$T' + \frac{a^2 z_n^2 T}{L^2} = 0$$

So $T_n(t) = d_n e^{\frac{-a^2 z_n^2 t}{L^2}}$

For each positive integer n, let

$$u_n(x,t) = c_n \sin\left(\frac{z_n x}{L}\right) e^{\frac{-a^2 z_n^2 t}{L^2}} \text{ where } c_n = a_n d_n$$

Each such function satisfies the heat equation and the boundary conditions. To satisfy the initial conditions, let

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{z_n x}{L}\right) e^{\frac{-a_n z_n^2 t}{L^2}}$$

we must choose the $c_n 's$ so that

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{z_n x}{L}\right) = f(x)$$

Unlike what we encountered in the other two examples, this is not a standard's Fourier series, because of the $z_n 's$. Indeed, we do not know these numbers, because they are solutions of a transcendental equation we cannot solve exactly.

At this point we must rely on the Strum- Liouville theorem, which states that the eigen functions of the Strum- Liouville problem are orthogonal on [0,L] with weight function 1. This means that if n and m are distinct positive integers, then

$$\int_0^L \sin\left(\frac{z_m x}{L}\right) \sin\left(\frac{z_n x}{L}\right) dx = 0$$

This is like the orthogonality relationship used to derive co-efficient of Fourier series and can be exploited in the same way to find the

$$c_n = \frac{\int_0^L f(x) \sin\left(\frac{z_n x}{L}\right) dx}{\int_0^L \sin^2\left(\frac{z_n x}{L}\right) dx}$$

With this choice of co-efficient, the solution is

$$u(x,t)=\sum_{n=1}^{\infty}\left[\frac{\int_0^L f(\xi)\sin\left(\frac{z_n\xi}{L}\right)d\xi}{\int_0^L \sin^2\left(\frac{z_n\xi}{L}\right)d\xi}\right]\sin\left(\frac{z_n x}{L}\right)e^{\frac{-a^2z_n^2t}{L^2}}$$

Problems:

1. A thin, homogeneous bar of thermal diffusivity 4 and length 6 cm with insulated sides, has its end maintained at temperature zero. Its right end is radiating (with transfer co-efficient $\frac{1}{2}$) into the surrounding medium, which has temperature zero. The bar has an initial temperature given by $f(x)=x(6-x)$. Approximate the temperature distribution $u(x,t)$ by finding the fourth partial sum of the series representation for $u(x,t)$.

1.2.4 Heat Conduction in a Bar with Ends at Different Temperature

Consider a thin, homogeneous bar extending from $x=0$ to $x=L$. The left end is maintained at constant temperature T_1 and the right end at constant temperature T_2. The initial temperature throughout the bar in the cross-section at x is $f(x)$.

The boundary value problem for the temperature distribution is:

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}\ (0<x<L\ ,\ t>0)$$
$$u(0,t)=T_1\ ,\ u(L,t)=T_2\ (t>0)$$
$$u(x,0)=f(x)\ (0\le x\le L)$$

Put $u(x,t)=X(x)T(t)$ into the heat equation to obtain,

$$X''+\lambda X=0$$
$$T'+\lambda a^2T=0$$

Unlike the preceding example, there is nothing in this partial differential equation that prevents separation of the variables. The difficulty encountered here is with the boundary conditions which are non-homogeneous ($u(0,t)$ and $u(L,t)$ may be non-zero). To see the effect of this consider, $u(0,t)=X(0)T(t)=T_1$

If $T_1=0$, we could conclude that $X(0)=0$. But if $T_1\neq 0$, this equation forces us to conclude that $T(t)=\frac{T_1}{X(0)}=\text{constant}$. This is a condition, we cannot except to satisfy. The boundary condition at L possess the same problem.

We attempt to eliminate the problem by perturbing the function. Set

$$u(x,t)=U(x,t)+\psi(x)$$

We want to choose $\psi(x)$ to obtain a problem, we can solve.

Substitute $u(x,t)$ into the partial differential equation to get

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + a^2 \psi''(x)$$

We obtain the heat equation for U if $\psi''(x) = 0$. Integrating twice, $\psi(x)$ must have the form

$$\psi(x) = Cx + D \qquad ...(1)$$

Now, consider the boundary conditions, first

$$u(0,t) = T_1 = U(0,t) + \psi(0)$$

This condition becomes $U(0,t) = 0$ if we choose $\psi(x)$ so that

$$\psi(0) = T_1 \qquad ...(2)$$

The condition

$$u(L,t) = T_2 = U(L,t) + \psi(L)$$

becomes $U(L,t) = 0$ if

$$\psi(L) = T_2 \qquad ...(3)$$

Now, use (2) and (3) to solve for C and D in (1),

$$\psi(0) = D = T_1$$

and $\psi(L) = CL + T_1 = T_2 \Rightarrow C = \frac{1}{L}(T_2 - T_1)$

Thus, choose

$$\psi(x) = \frac{1}{L}(T_2 - T_1)x + T_1$$

with this choice, the boundary value problem for $U(x,t)$ is

$$\frac{\partial U}{\partial t} = a^2 \frac{\partial^2 U}{\partial x^2}$$

$$U(0,t) = U(L,t) = 0$$

$$U(x,0) = u(x,0) - \psi(x) = f(x) - \frac{1}{L}(T_2 - T_1)x - T_1$$

We have solved this problem earlier, with the solution

$$U(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L \left[f(\xi) - \frac{1}{L}(T_2 - T_1)x + T \right]\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right)\sin\left(\frac{n\pi x}{L}\right)e^{\frac{-n^2\pi^2 a^2 t}{L^2}}$$

Once, we know this function, then

$$u(x,t) = U(x,t) + \frac{1}{L}(T_2 - T_1)x + T_1$$

1.3 Steady–State Temperature in Plates

The two-dimensional Heat equation is

$$\frac{\partial u}{\partial t} = a^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = a^2\nabla^2 u$$

The steady-state case occurs when we set $\frac{\partial u}{\partial t} = 0$. In this event, the Heat equation is Laplace's equation $\nabla^2 u = 0$

A Dirichlet problem consists of Laplace's equation, to be solved for (x,y) in a region R of the plane, together with prescribed values the solution is to assumes on the boundary of R, which is usually a piecewise smooth curve. If we think of R as a flat plate, then we are finding the steady-state temperature distribution throughout a plate, given the temperature at all timers on its boundary.

1.3.1 Steady-State Temperature in a Rectangular Plate

Consider a flat rectangular plate occupying the region R in the xy-plane by $0 \le x \le a,\ 0 \le y \le b.$ Suppose the right side is kept at constant temperature T, while the other sides are kept at temperature zero. The boundary value problem for the steady-state temperature distribution is:

$$\begin{aligned}
&\nabla^2 u = 0 \ (0 < x < a, 0 < y < b)\\
&u(x,0) = u(x,b) = 0 \ (0 < x < a)\\
&u(0,y) = 0 \quad (0 < y < b)\\
&u(a,y) = T \quad (0 < y < b)
\end{aligned}$$

Put $u(x,y) = X(x)Y(y)$ into Laplace's equation to get

$$X''Y + Y''X = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

Since the left side depends only on x and the right side only on y, and these variables are independent, both sides must equal the same constant.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda \text{ (say)}$$

Now, use the boundary condition:

$$\begin{aligned}
&u(x,0) = X(x)Y(0) = 0 \Rightarrow Y(0) = 0\\
&u(x,b) = X(x)Y(b) = 0 \Rightarrow Y(b) = 0
\end{aligned}$$

and $u(0,y) = X(0)Y(y) = 0 \Rightarrow X(0) = 0$

Therefore, $X(x)$ must satisfy

$$X''-\lambda X=0$$
$$X(0)=0$$

and, Y must satisfy

$$Y''+\lambda Y=0$$
$$Y(0)=Y(b)=0$$

This problem for $Y(y)$ was solved in the article (Ends of the bar kept at temperature zero) with $X(x)$ in place of $Y(y)$ and L in place of b.

The eigen values are

$$\lambda_n=\frac{n^2\pi^2}{b^2}$$

with corresponding eigen functions

$$Y_n(y)=b_n\sin\left(\frac{n\pi y}{b}\right) \text{ for n=1,2,...}$$

The problems for X is now

$$X''-\frac{n^2\pi^2}{b^2}X=0$$
$$X(0)=0$$

The general solution of the differential equation is

$$X_n(x)=ce^{\frac{n\pi x}{b}}+de^{\frac{-n\pi x}{b}}$$

Since $X(0)=c+d=0\Rightarrow d=-c$ and so

$$X_n(x)=c\left[e^{\frac{n\pi x}{b}}-e^{\frac{-n\pi x}{b}}\right]=2c_n\sinh\left(\frac{n\pi x}{b}\right)$$

For each positive integer n, let

$$u_n(x,y)=a_n\sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)\text{ ; where } a_n=2b_nc_n$$

For each n and any choice of the constant a_n this function satisfies Laplace’s equation and the zero boundary conditions on three sides of the plate. For the non-zero boundary condition, we must use a superposition

$$u(a,y)=T=\sum_{n=1}^{\infty}a_n\sin\left(\frac{n\pi y}{b}\right)\sinh\left(\frac{n\pi a}{b}\right)$$

This is a Fourier sine expansion of T on [a,b]. Therefore, choose the entire co-efficient

$\sin\left(\frac{n\pi y}{b}\right)$ as the Fourier sine co-efficient:

$$a_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b T \sin\left(\frac{n\pi y}{b}\right) dy$$
$$= \frac{2T}{b}\left[1-(-1)^n\right]\frac{b}{n\pi},$$

in which we have used the fact that $\cos n\pi = (-1)^n$, if n is an integer.

We now have

$$a_n = \frac{2T}{n\pi}\frac{1}{\sinh\left(\frac{n\pi a}{b}\right)}\left[1-(-1)^n\right]$$

The solution is

$$u(x,y) = \frac{2T}{\pi}\sum_{n=1}^{\infty}\frac{1}{n\sinh\left(\frac{n\pi a}{b}\right)}\left[1-(-1)^n\right]\sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

As we have done before, observe that $1-(-1)^n$ equals 0 if n is even, and equals 2 if n is odd. We can therefore omit the even indices in this summation, writing the solution as:

$$u(x,y) = \frac{4T}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\sinh\left(\frac{(2n-1)\pi a}{b}\right)}\sinh\left(\frac{(2n-1)\pi x}{b}\right)\sin\left(\frac{(2n-1)\pi y}{b}\right)$$

Problems: 1. Solve for the steady-state temperature distribution in a flat plate covering the region $0 \le x \le a,\ 0 \le y \le b,$ if the temperature on the vertical sides and the bottom side are kept at zero while the temperature on the top side is a constant K.

2.Solve for the steady-state temperature distribution is a flat plate covering the region $0 \le x \le a,\ 0 \le y \le b,$ if the temperature on the left side is a constant T_1 and that on right side a constant T_2, while the top and bottom sides are kept at temperature zero.

[Hint: Consider two separate problems. In the first, the temperature on the left side is T_1 and the other sides are kept at temperature zero. In the second, the temperature on the right side is T_2, while the other sides are kept at zero. The sum of solutions of these problems is the solution of the original problem.]

Remark: It is possible to treat the case where the four sides are kept at different temperature (not necessarily constant), by considering four plates, in each of which the temperature is non-zero on only one side of the plate. The sum of the solutions of these four problems is the solution for the original plate.

1.3.2 Steady-State Temperature in a Circular Disc

Consider a thin disk of radius R, placed in the plane so that its centre is the origin. We will find the steady-state temperature distribution $u(r,\theta)$ as a function of polar co-ordinates. The Laplace's equation in polar co-ordinates is

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$$

for $0 \le r \le R$ and $-\pi \le \theta \le \pi$

Assume that the temperature is known on the boundary of the disk:

$$u(R,\theta)=f(\theta) \qquad \text{for} \qquad -\pi \le \theta \le \pi$$

In order to determine a unique solution for u, we will specify two additional conditions, First we seek a bounded solution. This is certainly a physically reasonable condition. Second we assume periodically conditions:

$$u(r,\pi)=u(r,-\pi) \quad \text{and} \quad \frac{\partial u}{\partial \theta}(r,\pi)=\frac{\partial u}{\partial \theta}(r,-\pi)$$

These conditions account for the fact that (r,π) and $(r,-\pi)$ are polar co-ordinates of the same point.

Attempt a solution

$$u(r,\theta)=F(r)G(\theta)$$

Substitute this into the Laplace's equation, we get

$$F''(r)G(\theta)+\frac{1}{r}F'(r)G(\theta)+\frac{1}{r^2}F(r)G''(\theta)=0$$

If $F(r)G(\theta)\neq 0$, this equation can be written

$$\frac{r^2F''(r)+rF'(r)}{F(r)}=-\frac{G''(\theta)}{G(\theta)}$$

Since the left side of this equation depends only on r and the right side only on θ, and these variables are independent, both sides must equal same constant

$$\frac{r^2F''(r)+rF'(r)}{F(r)}=-\frac{G''(\theta)}{G(\theta)}=\lambda$$

which gives

$$r^2F''(r)+rF'(r)-\lambda F(r)=0 \qquad ...(1)$$
$$\text{and } G''(\theta)+\lambda G(\theta)=0$$

Now, consider the boundary conditions. First

$$u(r,\pi)=u(r,-\pi)\Rightarrow G(\pi)F(r)=G(-\pi)F(r)$$

Assuming $F(r)$ is not identically zero, then

$$G(\pi)=G(-\pi)$$

Similarly,

$$\frac{\partial u}{\partial\theta}(r,\pi)=F(r)G'(\pi)=\frac{\partial u}{\partial\theta}(r,-\pi)=F(r)G'(-\pi)$$
$$\Rightarrow G'(\pi)=G'(-\pi)$$

The problem to solve $G(\theta)$ is therefore

$$\left.\begin{array}{l} G''(\theta)+\lambda G(\theta)=0 \\ G(\pi)=G(-\pi) \\ G'(\pi)=G'(-\pi) \end{array}\right\} \qquad ...(2)$$

This is a periodic Strum-Liouville problem and first we solve it by considering different cases:

Case 1: $\lambda=0$

In this case, the equation reduces to

$$G''(\theta)=0$$

with the general solution

$$G(\theta)=c+d\theta$$

Now,

$$G(\pi)=G(-\pi)\Rightarrow c+d\pi=c-d\pi\Rightarrow 2d\pi=0$$
$$\Rightarrow d=0$$
$$\therefore G(\theta)=c$$

which satisfies $G'(\pi)=G'(-\pi)$

Thus, $\lambda=0$ is an eigen value of the problem with eigen function

$$G(\theta)=c_0=\text{constant}$$

Case 2: $\lambda<0$

Let $\lambda=-n^2$

Then, the differential equation (2) is

$$G''(\theta) - n^2 G(\theta) = 0$$

with the general solution given by

$$G(\theta) = ce^{n\theta} + de^{-n\theta}$$

Now,

$$G(\pi) = G(-\pi) \Rightarrow ce^{n\pi} + de^{-n\pi} = ce^{-n\pi} + de^{n\pi}$$
$$\therefore G(\theta) = c\left(e^{n\theta} + e^{-n\theta}\right) \Rightarrow c - d = 0 \Rightarrow c = d$$

Also,

$$G'(\pi) = G'(-\pi) \Rightarrow cn(e^{n\pi} - e^{-n\pi}) = cn(e^{-n\pi} - e^{n\pi})$$
$$\Rightarrow 2cn = 0 \Rightarrow c = 0$$
$$\therefore G(\theta) = 0$$

Thus, we have no eigen value in this case.

Case 3: $\lambda > 0$

Let $\lambda = k^2$. Then, the differential equation (2) is

$$G''(\theta) + k^2 G(\theta) = 0$$

with the general solution given by

$$G(\theta) = c\cos(k\theta) + d\sin(k\theta)$$

Now,

$$G(\pi) = G(-\pi) \Rightarrow c\cos(k\pi) + d\sin(k\pi) = c\cos(k\pi) - d\sin(k\pi)$$
$$\Rightarrow 2d\sin(k\pi) = 0$$

For a non-trivial solution, we take

$$k\pi = n\pi \quad \text{for n=1,2...}$$
$$\Rightarrow k = n \quad \text{for n=1,2...}$$

Similarly, result holds for $G'(\pi) = G'(-\pi)$

Thus, the general solution is given by

$$G_n(\theta) = c_n\cos(n\theta) + d_n\sin(n\theta)$$

Thus, the eigen values for the SLBVP (2) is

$$\lambda = n^2 \text{ ; n=0,1,2,3...}$$

and the eigen function is

$$G_0(\theta) = c_0$$
$$G_n(\theta) = c_n \cos(n\theta) + d_n \sin(n\theta)$$

Now, let $\lambda = n^2$ to get (1) as

$$rF''(r) + rF'(r) - n^2 F(r) = 0$$

This is a second order Euler differential equation with general solution

$$F_n(r) = a_n r^n + b_n r^{-n} \quad , \text{ for n=1,2,3...}$$
and $F_0(r) = a_0 = \text{constant}$, for n=0

The requirement that the solution must be bounded forces to choose each $b_n = 0$ because $r^{-n} \to \infty$ as $r \to 0^+$ (centre of the disk).

Combining cases, we can write

$$F_n(r) = a_n r^n \text{ for n=0,1,2...}$$

For n=0,1,2…, we now have functions of the form

$$u_n(r,\theta) = F_n(r) G_n(\theta) = a_n r^n \left[c_n \cos(n\theta) + d_n \sin(n\theta) \right]$$

Setting $A_n = a_n c_n$ and $B_n = a_n d_n$, we have

$$u_n(r,\theta) = r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta) \right]$$

These functions satisfy Laplace's equation and the periodicity conditions, as well as the condition that solutions must be bounded. For any given n, this function will generally not satisfy the initial condition

$$u(R,\theta) = f(\theta)$$

For this, use the superposition

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta) \right]$$

Now, the initial condition requires that

$$u(R,\theta) = f(\theta) = A_0 + \sum_{n=1}^{\infty} \left[A_n R^n \cos(n\theta) + B_n R^n \sin(n\theta) \right]$$

This is the Fourier series expansion of $f(\theta)$ on $[-\pi, \pi]$. Thus, choose

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

$$A_n R^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta \Rightarrow A_n = \frac{1}{\pi R^n} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta$$

and $\mathrm{B}_n R^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta \Rightarrow B_n = \frac{1}{\pi R^n} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta$

Example: As a specific example, suppose the disk has radius 3 and that $f(\theta)=2+\theta$. A routine integration gives

$$A_0=2\ ,\ A_n=0 \text{ for } n=1,2,3...$$

$$\text{and } B_n=\frac{2}{n.3^n}(-1)^{n+1}$$

The solution for this condition is

$$u(r,\theta)=2+\sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\left(\frac{r}{3}\right)^n \sin(n\theta)$$

for $0\le r\le 3$ and $-\pi\le\theta\le\pi$.

Problems:

1. Find the steady-state temperature for a thin disk

i. of radius R with temperature on boundary is $f(\theta)=\cos^2\theta$ for $-\pi\le\theta\le\pi$

ii. of radius 1 with temperature on boundary is $f(\theta)=\cos^3\theta$ for $-\pi\le\theta\le\pi$

iii. of radius R with temperature on boundary is constant T.

2. Use the solution of steady-state temperature distribution in a thin disk to show that the temperature at the centre of disk is the average of the temperature values on the circumference of the disk.

[Hint: For temperature on the centre of disk, we let $r\to 0^+$, so that $u(r,\theta)=A_0=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)d\theta$

which is the average of $f(\theta)$, the temperature on the circumference of the disk.]

3. Find the steady-state temperature in the flat wedge-shaped plate occupying the region $0\le r\le k$, $0\le\theta\le\alpha$ (in polar co-ordinates). The sides $\theta=0$ and $\theta=\alpha$ are kept at temperature zero and the ark $r=k$ for $0\le\theta\le\alpha$ is kept at temperature T.

[Hint: The BVP for this situation is

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0$$

$$u(r,0)=u(r,\alpha)=0\ \ (0\le r\le k)$$

$$u(k,\theta)=T\ \ (0\le\theta\le\alpha)$$

1.3.3 Steady-State Temperature Distribution in a Semi-infinite Strip

Find the steady-state temperature distribution in a semi-infinite strip $x\ge 0,\ 0\le y\le 1,$ pictured in figure. The temperature on the top side and bottom side are kept at zero, while the left side is kept at temperature T.

The boundary value problem modelling this problem is:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad (0\le y\le 1, x\ge 0)$$
$$u(x,0)=0=u(x,1) \quad (\mathrm{x}\ge 0)$$
$$\mathrm{u(0,y)=T} \qquad (0\le y\le 1)$$

Put $u(x,y)=X(x)Y(y)$ into Laplace's equation to get

$$X''Y+XY''=0 \;\Rightarrow\; \frac{X''}{X}=\frac{-Y''}{Y}$$

Since the left side depends only on x and right side only on x, and these variables are independent, both sides must equal the same constant:

$$\frac{X''}{X}=\frac{-Y''}{Y}=\lambda$$

Now, use the boundary conditions:

$$u(x,0)=X(x)Y(0)=0\Rightarrow Y(0)=0$$
$$u(x,1)=X(x)Y(1)=0\Rightarrow y(1)=0$$

Therefore, X must satisfy

$$X''-\lambda X=0$$

and, Y must satisfy

$$Y''+\lambda Y=0$$
$$Y(0)=Y(1)=0$$

The solution for the equation for $Y(y)$ is given by (by above article)

$$Y_n(y)=a_n\sin(n\pi y) \qquad \text{for n=1,2,...}$$

with the eigen value given by

$$\lambda_n=n^2\pi^2$$

The problem for $X(x)$ is now

$$X''-n^2\pi^2X=0$$

The general solution of the differential equation is

$$X_n(x)=b_ne^{n\pi x}+c_ne^{-n\pi x}$$

Now, since $u(x,y)<\infty$, so $b_n=0$, otherwise $X_n(x)\to\infty$ as $\mathrm{x}\to\infty$. Thus, we have

$$X_n(x)=c_ne^{-n\pi x}$$

Thus, solution for each n is

$$u_n(x,y) = d_n e^{-n\pi x}\sin(n\pi y) \text{ , where } d_n = a_n c_n$$

For each n, using the superposition, we have

$$u(x,y) = \sum_{n=1}^{\infty} d_n e^{-n\pi x}\sin(n\pi y)$$

We want to choose the constant d_n, so that

$$u(0,y) = T = \sum_{n=1}^{\infty} d_n \sin(n\pi y)$$

which is Fourier sine expansion of T on [0,1]. Therefore, choose the entire co-efficient of $\sin(n\pi y)$ as the Fourier sine co-efficient:

$$\begin{aligned} d_n &= 2\int_0^1 T\sin(n\pi y)dy \qquad \text{[As in above article]} \\ &= \frac{2T}{n\pi}\left[1-(-1)^n\right] \end{aligned}$$

Problem:

1. Find a steady-state temperature distribution in the semi-infinite region $0 \le x \le a, y \ge 0$ if the temperature on the bottom and left sides are at zero and the temperature on the right side is kept at constant T.

2. Find the steady-state temperature distribution in the semi-infinite region $0 \le x \le 4, y \ge 0$ if the temperature on the vertical sides are kept at constant T and temperature on the bottom side is kept at zero.

   [Hint: Assume two semi-infinite regions, first with left end at temperature T and right end and bottom at temperature zero, second with right end at temperature zero and left end and bottom at temperature zero. Sum of these two solutions is the solution of the original problem.]

3. Use your intuition to guess the steady-state temperature in a thin rod of length L if the ends are perfectly insulated and the initial temperature is $f(x)$ for $0 < x < L.$

   [Hint: The boundary value problem modelling this problem is

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0 \qquad (0 < x < L)\ (t > 0) \\ &\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t) \qquad (t > 0) \\ &u(x,0) = f(x) \qquad (0 < x < L) \end{aligned}$$

1.3.4 Steady-State Temperature in a Semi-infinite Plate

The B.V.P. is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad (0\le y\le b,\quad x\ge 0)$$
$$u(x,0)=0=u(x,b) \qquad (x\ge 0)$$
$$u(0,y)=T \qquad (0\le y\le b)$$

Put $u(x,y)=X(x)Y(y)$ into the given Laplace equation, we obtain

$$X''Y+XY''=0 \Rightarrow \frac{X''}{X}=-\frac{Y''}{Y}$$

Since the left side is depend on x only while right hand side is y only. So both side must be equal to some constant. Let the constant of separation coefficient is λ . The above equation becomes

$X''-\lambda X=0$ and $Y''+\lambda Y=0$

And the boundary condition $\begin{array}{l} u(x,0)=X(x)Y(0)=0\Rightarrow Y(0)=0 \\ u(x,b)=X(x)Y(b)=0\Rightarrow Y(b)=0 \end{array}$

Here we have more information for problem Y with equations

$$Y''+\lambda=0$$
$$Y(0)=0 \quad and \quad Y(b)=0$$

In earlier article, we solve such problem and preceding like that, we have solution

$$Y_n=a_n\sin\left(\frac{n\pi y}{b}\right) \quad for \;\; n=1,2,3...$$

with the eigen value $\lambda_n=\dfrac{n^2\pi^2}{b^2}$.

Now the problem for X is

$$X''-\frac{n^2\pi^2}{b^2}X=0$$

The general solution is

$$X_n(x)=b_n e^{\frac{n\pi x}{b}}+c_n e^{-\frac{n\pi x}{b}}$$

For a bounded solution in the given domain, we have to assume $b_n=0$. Now the solution becomes $X_n(x)=c_n e^{-\frac{n\pi x}{b}}$. Thus the solution for each n by using the superposition is

$$u(x,y)=\sum_{n=1}^{\infty} d_n e^{-\frac{n\pi x}{b}} \sin\left(\frac{n\pi y}{b}\right) \; where \; d_n = a_n c_n$$

Now using the condition $u(0,T)=T$, we have

$u(0,y)=T=\sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi y}{b}\right)$ which is a Fourier sine expansion of T on [0,1]. The coefficient

$$d_n = 2\int_0^b T\sin\left(\frac{n\pi y}{b}\right)dy$$

$$=\frac{2T}{n\pi}\left[1-(-1)^n\right]$$

$$u(x,y)=\sum_{n=1}^{\infty}=\frac{2T}{n\pi}\left[1-(-1)^n\right]\sin\left(\frac{n\pi y}{b}\right)e^{\frac{-n\pi x}{b}}$$

1.3.5 Steady-State Temperature in an Infinite Plate

Suppose we want the steady-state temperature distribution in a thin, flat plate extending over the right quarter plane $x \geq 0, y \geq 0$. Assume that the temperature on the vertical side $x=0$ is kept at zero, while the bottom side $y=0$ is kept at a temperature $f(x)$.

The BVP modelling this problem is:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad (x>0, y>0)$$
$$u(0,y)=0 \quad (y>0)$$
$$u(x,0)=f(x) \quad (x>0)$$

Now solving as in previous examples we get

$$u(x,y)=\int_0^{\infty} c_k \sin(kx)e^{-ky}dk$$

Finally, we require that

$$u(x,0)=f(x)=\int_0^{\infty} c_k \sin(kx)dk$$

This is the Fourier sine integral of $f(x)$ on $[0,\infty)$, so choose

$$c_k=\frac{2}{\pi}\int_0^{\infty} f(\xi)\sin(k\xi)d\xi$$

Thus, the solution for the problem is

$$u(x,y)=\int_0^\infty\left[\frac{2}{\pi}\int_0^\infty f(\xi)\sin(k\xi)d\xi\right]\sin(kx)e^{-ky}dk$$

Example: Assume that in the above problem

$$f(x)=\begin{cases}4\ , & 0\le x\le 2\\ 0\ , & x>2\end{cases}$$

Then,

$$\begin{aligned}c_k&=\frac{2}{\pi}\int_0^\infty f(\xi)\sin(k\xi)d\xi\\&=\frac{2}{\pi}\int_0^\infty 4\sin(k\xi)d\xi\\&=\frac{8}{\pi k}\left[1-\cos(2k)\right]\end{aligned}$$

Thus,

$$u(x,y)=\frac{8}{\pi}\int_0^\infty\left[\frac{1-\cos 2k}{k}\right]\sin(kx)e^{-ky}dk$$

1.4 Heat Equation in Unbounded Domains

Here, we will discuss the problems of temperature distribution in a bar with the space variable extending over the real line or half line.

1.4.1 Heat Conduction in a Semi-Infinite Bar

Suppose we want the temperature distribution in a Bar stretching from 0 to ∞ along the x-axis. The left end is kept at temperature zero and the initial temperature in the cross-section at x is $f(x)$.

The boundary value problem for the temperature distribution is:

$$\begin{aligned}&\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}\qquad (x>0,t>0)\\&u(x,0)=f(x)\quad (x>0)\\&u(0,t)=0\qquad (t>0)\end{aligned}$$

As usual, we seek a solution, which is bounded.

Set,

$u(x,t)=X(x)T(t)$ to get

$$X''+\lambda X=0\ \ (x>0)$$

$$T'+\lambda a^2T=0\ \ (t>0)$$

Now as in previous examples, we get

$$u(x,t) = d_k \sin(kx) e^{-a^2k^2t}, \text{ where } d_k = a_k b_k$$

Now, using the superposition

$$u(x,t) = \int_0^\infty d_k \sin(kx) e^{-a^2k^2t} dk \qquad ...(1)$$

Finally, we must satisfy the initial condition:

$$f(x) = u(x,0) = \int_0^\infty d_k \sin(kx) dk \qquad ...(2)$$

For this choice the d_k's are the Fourier sine integral co-efficient of $f(x)$; so

$$d_k = \frac{2}{\pi}\int_0^\infty f(\xi)\sin(k\xi)d\xi$$

With this choice of the co-efficient, the function defined by (1) is a solution of the problem.

Example: Suppose

$$f(x) = \begin{cases} \pi - x \ , & 0 \le \mathrm{x} \le \pi \\ 0 \ , & x > \pi \end{cases}$$

Then, $d_k = \dfrac{2}{\pi}\displaystyle\int_0^\pi (\pi - \xi)\sin(k\xi)d\xi = \dfrac{2}{k}\left(1 - \dfrac{\sin(k\pi)}{k\pi}\right)$

The solution is

$$u(x,t) = \int_0^\infty \frac{2}{k}\left(1 - \frac{\sin(k\pi)}{k\pi}\right)\sin(kx) e^{-k^2\pi^2t} dk$$

1.4.2 Heat Conduction in Infinite Bar

Suppose we want the temperature distribution in a Bar stretching from $-\infty$ to ∞ along the x-axis. The initial temperature in the cross-section at x is $f(x)$. The boundary value problem for the temperature distribution is:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (-\infty < x < \infty, t > 0)$$

$$u(x,0) = f(x) \quad (-\infty < x < \infty)$$

There are no boundary conditions, so we impose the physically realistic condition that solutions should be bounded. As usual, we seek a solution, which is bounded.

Set,

$u(x,t) = X(x)T(t)$ to get

$X'' + \lambda X = 0 \ \ (-\infty < x < \infty)$

$T' + \lambda a^2 T = 0 \ (t > 0)$

Now as in previous examples, we get

$$u_k(x,t) = (a_k \cos(kx) + b_k \sin(kx))e^{-a^2k^2t},$$

that satisfy the Heat equation and are bounded on the real line over all k>0. Now, using the superposition

$$u(x,t) = \int_0^\infty (a_k \cos(kx) + b_k \sin(kx))e^{-a^2k^2t} dk \qquad ...(1)$$

Finally, we must satisfy the initial condition:

$$f(x) = u(x,0) = \int_0^\infty (a_k \cos(kx) + b_k \sin(kx)) dk \qquad ...(2)$$

For this choice the a_k's and b_k's are the Fourier sine integral co-efficient of $f(x)$; so

$$a_k = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(k\xi)d\xi$$

and

$$a_k = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(k\xi)d\xi$$

With this choice of the co-efficient, the function defined by (1) is a solution of the problem.

1.5 Solution of Heat, Laplace and Wave Equations

1.5.1 Solution of Three-Dimensional Heat Equations in Cartesian co-ordinates

It is a partial differential equation of the form:

$$\frac{\partial u}{\partial t} = h^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \qquad ...(1)$$

To find its solution by the method of separation of variables, suppose that the solution of (1) is

$$u(x,y,z,t) = X(x)Y(y)Z(z)T(t) \qquad ...(2)$$

where $X(x)$ is a function of x only, $Y(y)$ is a function of y only, $Z(z)$ is a function of z only and $T(t)$ is a function of t only.

We get on separating the variables

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Y}\frac{d^2Z}{dz^2} = \frac{1}{h^2T}\frac{dT}{dt} \qquad ...(3)$$

Choosing the constant of separation such that

$$\frac{1}{X}\frac{d^2X}{dx^2}=-p_1^2,\frac{1}{Y}\frac{d^2Y}{dy^2}=-p_2^2,\frac{1}{Z}\frac{d^2Z}{dz^2}=-p_3^2 \text{ and } \frac{1}{h^2T}\frac{dT}{dt}=-p^2,\text{where } p^2=p_1^2+p_2^2+p_3^2$$

Thus, we have the following three equations

$$\frac{d^2X}{dx^2}+p_1^2X=0$$
$$\frac{d^2Y}{dy^2}+p_2^2Y=0$$
$$\frac{d^2Z}{dz^2}+p_3^2Z=0$$
$$\frac{dT}{dt}+p^2h^2T=0$$

with the solutions

$$X(x)=A\cos p_1x+B\sin p_1x$$
$$Y(y)=C\cos p_2y+D\sin p_2y$$
$$Y(y)=E\cos p_3z+F\sin p_3z$$
$$T(t)=Ge^{-p^2h^2t}=Ge^{-(p_1^2+p_2^2+p_3^2)h^2t}$$

Combining these solutions and using the superposition, we get

$$u(x,y,z,t)=\sum_{p_1,p_2,p_3=1}^{\infty}(A\cos p_1x+B\sin p_1x)(C\cos p_2y+D\sin p_2y)(E\cos p_3z+F\sin p_3z)Ge^{-(p_1^2+p_2^2+p_3^2)h^2t}$$

Corollary: The Heat equation in two-dimensional is

$$\frac{\partial u}{\partial t}=h^2\left[\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right]$$

The solution is

$$u(x,y,t)=\sum_{p_1,p_2=1}^{\infty}(A\cos p_1x+B\sin p_1x)(C\cos p_2y+D\sin p_2y)Ee^{-(p_1^2+p_2^2)h^2t}$$

1.5.2 Solution of Heat Equation in Cylindrical Polar Co-ordinates

In cylindrical co-ordinates, Heat equation has the form

$$\frac{\partial^2u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}+\frac{\partial^2u}{\partial z^2}=\frac{1}{h^2}\frac{\partial u}{\partial t} \quad ...(1)$$

To solve it by the method by separation of variables, we have

$$u(r,\theta,z,t)=R(r)\Theta(\theta)Z(z)T(t) \quad ...(2)$$

giving

$$\frac{\partial u}{\partial r}=\frac{dR}{dr}\Theta(\theta)Z(z)T(t) \quad , \quad \frac{\partial^2 u}{\partial r^2}=\frac{d^2R}{dr^2}\Theta(\theta)Z(z)T(t)$$

$$\frac{\partial^2 u}{\partial \theta^2}=R(r)\frac{d^2\Theta}{d\theta^2}Z(z)T(t) \quad , \quad \frac{\partial^2 u}{\partial z^2}=R(r)\Theta(\theta)\frac{d^2Z}{dz^2}T(t)$$

$$\frac{\partial u}{\partial t}=R(r)\Theta(\theta)Z(z)\frac{dT}{dt}$$

Substituting all these values in equation (1), we get

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+\frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2}+\frac{d^2Z}{dz^2}=\frac{1}{h^2T}\frac{dT}{dt}$$

Using the method of separation of variables, we have

$$\frac{1}{h^2T}\frac{dT}{dt}=-\lambda^2 \Rightarrow \frac{dT}{dt}+\lambda^2h^2t=0 \qquad (3)$$

$$\frac{d^2Z}{dz^2}=-\kappa^2 \Rightarrow \frac{d^2Z}{dz^2}+\kappa^2h^2t=0 \qquad (4)$$

and $$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=-\mu^2 \Rightarrow \frac{d^2\Theta}{d\theta^2}+\mu^2\Theta=0 \qquad ...(5)$$

so that

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)-\frac{\mu^2}{r^2}-\kappa^2=-\lambda^2$$

$$\Rightarrow \frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(\varepsilon^2-\frac{\mu^2}{r^2}\right)R=0 \qquad \text{where } \varepsilon^2=\lambda^2\text{-}\kappa^2 \qquad ...(6)$$

with solution of (3) as

$$T(t)=ae^{-h^2\lambda^2t}$$

solution of (4) as

$$Z(z)=b\cos(\kappa z)+c\sin(\kappa z)$$

solution of (5) as

$$\Theta(\theta)=e\cos(\mu\theta)+f\sin(\mu\theta)$$

The equation (6) is Modified Bessel's Equation and the solution is

$$R(r)=AJ_{\mu}(\varepsilon r)+BJ_{-\mu}(\varepsilon r) \qquad \text{for fractional } \mu$$

and

$$R(r)=AJ_{\mu}(\varepsilon r)+BY_{-\mu}(\varepsilon r) \qquad \text{for integral } \mu$$

where

$$J_n(x)=\left(\frac{x}{2}\right)^n\sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{x}{2}\right)^{2r}}{(n+1)_r}\quad,\quad \text{where } (n+1)_r=(n+1)(n+2)\ldots(n+r)$$

$$J_{-n}(x)=(-1)^n J_n(-x)$$

and

$$Y_n(x)=\frac{2}{\pi}\left\{\log\frac{x}{2}+\gamma\right\}J_n(x)-\frac{1}{\pi}\sum_{p=0}^{n-1}\frac{\overline{\lfloor n-p}}{\lfloor p}\left(\frac{2}{x}\right)^{n-2p}$$

Thus, the solution of Heat equation is

$$u(r,\theta,z,t)=\sum_{\lambda,\kappa,\mu}ae^{-h^2\lambda^2t}\left[b\cos(\mu\theta)+c\sin(\mu\theta)\right]\left[e\cos(\kappa z)+f\sin(\kappa z)\right]\left[AJ_\mu(\varepsilon r)+BJ_{-\mu}(\varepsilon r)\right]$$

for fractional μ,

and $$u(r,\theta,a,t)=\sum_{\lambda,\kappa,\mu}ae^{-h^2\lambda^2t}\left[b\cos(\mu\theta)+c\sin(\mu\theta)\right]\left[e\cos(\kappa z)+f\sin(\kappa z)\right]\left[AJ_\mu(\varepsilon r)+BY_\mu(\varepsilon r)\right]$$

for integral μ.

Corollary: In 2-dimesnion, the cylindrical heat Equation is

$$\frac{\partial^2u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}=\frac{1}{h^2}\frac{\partial u}{\partial t}$$

and the solution of Heat equation is

$$u(r,\theta,t)=\sum_{\lambda,\mu}ae^{-h^2\lambda^2t}\left[b\cos(\mu\theta)+c\sin(\mu\theta)\right]\left[AJ_\mu(\lambda r)+BJ_{-\mu}(\lambda r)\right]\text{ for fractional }\mu,$$

and $$u(r,\theta,t)=\sum_{\lambda,\mu}ae^{-h^2\lambda^2t}\left[b\cos(\mu\theta)+c\sin(\mu\theta)\right]\left[AJ_\mu(\lambda r)+BY_\mu(\lambda r)\right]\text{ for integral }\mu.$$

1.5.3 Solution of Heat Equation in Spherical Co-ordinates

In spherical polar co-ordinates, it has the form

$$\frac{\partial^2u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2u}{\partial\phi^2}=\frac{1}{h^2}\frac{\partial u}{\partial t}\qquad\ldots(1)$$

Assuming $u(r,\theta,\phi,t)=R(r)\Theta(\theta)\Phi(\phi)T(t)$, equation (1) becomes

$$\left[\frac{1}{R}\frac{d^2R}{dr^2}+\frac{2}{Rr}\frac{dR}{dr}+\frac{1}{\Theta r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{r^2\sin^2\theta\Phi}\frac{d^2\Phi}{d\phi^2}\right]=\frac{1}{h^2}\frac{dT}{dt}$$

Let $$\frac{1}{h^2T}\frac{dT}{dT}=-\lambda^2\ \Rightarrow\ \frac{dT}{dt}+\lambda^2h^2T=0\qquad\ldots(2)$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-m^2\quad\Rightarrow\frac{d^2\Phi}{d\phi^2}+m^2\Phi=0\qquad\ldots(3)$$

with solution given by

$$T(t) = ae^{-\lambda^2 h^2 t}$$
$$\Theta(\phi) = be^{\pm im\phi}$$

and

$$\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -\left(\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr}\right) - \lambda^2 r^2 = n(n+1) \text{ (say)}$$

giving

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \left(\lambda^2 r^2 - n(n+1)\right)R = 0 \qquad ...(4)$$
$$\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0 \qquad ...(5)$$

Here (4), being homogeneous, if we put $r = e^s$ and $D \equiv \dfrac{d}{ds}$, reduces to

$$\left[D(D-1) + 2D - n(n+1)\right]R = 0$$
or $\quad (D-n)(D+(n+1))R = 0$
$$\therefore \quad R = Ae^{ns} + Be^{-(n+1)s}$$
$$= Ar^n + Br^{-n-1}$$

Putting $\mu = \cos\theta$ in (5), so that

$$\frac{d\Theta}{d\theta} = \frac{d\Theta}{d\mu}\frac{d\mu}{d\theta} = -\sin\theta\frac{d\Theta}{d\mu} \Rightarrow \frac{1}{\sin\theta}\frac{d}{d\theta} = -\frac{d}{d\mu}$$

we have

$$\frac{d}{d\mu}\left\{\left(1-\mu^2\right)\frac{d\Theta}{d\mu}\right\} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}\Theta = 0$$
or $(1-\mu^2)\dfrac{d^2\Theta}{d\mu^2} - 2\mu\dfrac{d\Theta}{d\mu} + \left\{n(n+1) - \dfrac{m^2}{1-\mu^2}\right\}\Theta = 0$

which is associated Legendre equation, then the solution is of the form

$$\Theta = \Theta(\cos\theta)$$

and hence solution of given problem is

$$u_n(r,\theta,\phi,t) = (Ar^n + Br^{-n-1})\Theta(\cos\theta)e^{\pm im\phi}e^{-\lambda^2 h^2 t}$$

Hence, summing overall n and trying superposition, the general solution of (1) may be expressed as

$$u(r,\theta,\phi,t) = \sum_{n,\lambda,m} (A_n r^n + \frac{B_n}{r^{n+1}})\Theta(\cos\theta)e^{\pm i\lambda\phi}e^{-\lambda^2 h^2 t}$$ is required solution.

1.5.4 Solution of Laplace Equation in Cartesian Co-ordinates

In Cartesian co-ordinates, the Laplace equation has the form

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad ...(1)$$

To solve it by the method of separation of variables, we have

$$V(x,y,z) = X(x)Y(y)Z(z) \qquad ...(1)$$

giving $\frac{\partial^2 V}{\partial x^2} = \frac{d^2 X}{dx^2}$ YZ, $\quad \frac{\partial^2 V}{\partial y^2} = X\frac{d^2 Y}{dy^2}$ Z and $\quad \frac{\partial^2 V}{\partial z^2} = XY\frac{d^2 Z}{dz^2}$

so that (1) gives

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -p_1^{\,2} \Rightarrow \frac{d^2 X}{dx^2} + p_1^{\,2} X = 0 \qquad ...(2)$$

$$\frac{1}{Y}\frac{d^2 Y}{dy^2} = -p_2^{\,2} \Rightarrow \frac{d^2 Y}{dy^2} + p_2^{\,2} Y = 0 \qquad ...(3)$$

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = p^2 \Rightarrow \frac{d^2 Z}{dz^2} - p^2 Z = 0 \qquad \text{where } p^2 = p_1^{\,2} + p_2^{\,2} \qquad ...(4)$$

The solutions of these equations are

$$X(x) = A\cos p_1 x + B\sin p_1 x$$

$$Y(y) = C\cos p_2 y + D\sin p_2 y$$

$$Z(z) = Ee^{pz} + Fe^{-pz}$$

The combined solution of (1) is

$$V_p(x,y,z) = (A\cos p_1 x + B\sin p_1 x)(C\cos p_2 y + D\sin p_2 y(ce^{pz} + De^{-pz})$$

Using the superposition, we have

$$V(x,y,z) = \sum_{p_1,p_2} (A\cos p_1 x + B\sin p_1 x)(C\cos p_2 y + D\sin p_2 y(ce^{pz} + De^{-pz})$$

Corollary: In 2-dimesnion, the Laplace equation has the form

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad ...(1)$$

To solve it by the method of separation of variables, we have

$$V(x,y) = X(x)Y(y) \qquad ...(2)$$

giving $\dfrac{\partial^2 V}{\partial x^2} = \dfrac{d^2 X}{dx^2}\mathrm{Y}$ and $\dfrac{\partial^2 V}{\partial y^2} = X\dfrac{d^2 Y}{dy^2}$

so that (1) gives

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -\frac{1}{Y}\frac{d^2 Y}{dy^2} = -p^2$$

Now,

$$\frac{d^2 X}{dx^2} + p^2 X = 0 \qquad \text{...(3)}$$

and $$\frac{d^2 Y}{dy^2} - p^2 Y = 0 \qquad \text{...(4)}$$

The solutions of these equations are

$$X(x) = A\cos px + B\sin px$$
$$Y(y) = Ce^{py} + De^{-py}$$

The combined solution of (1) is

$$V_p(x, y) = (A\cos px + B\sin px)(ce^{py} + De^{-py})$$

Using the superposition, we have

$$V(x, y) = \sum_p [(A\cos px + B\sin px)(ce^{py} + De^{-py})]$$

1.5.5 Solution of Three-Dimensional Laplace Equation in Cylindrical Co-ordinates

In cylindrical co-ordinates, Laplace's equation has the form

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{...(1)}$$

Assuming that $V(r, \theta, z) = R(r)\Theta(\theta)Z(z)$, then (1) yields

$$\frac{1}{R}\frac{d^2 R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0 \qquad \text{...(2)}$$

Since the variables are separated, we can take

$$\frac{1}{z}\frac{\partial^2 Z}{\partial z^2} = \lambda^2 \quad \text{and} \quad \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -\mu^2$$

$$\Rightarrow \frac{\partial^2 Z}{\partial z^2} - \lambda^2 z = 0 \quad \text{and} \quad \frac{d^2\Theta}{d\theta^2} + \mu^2\Theta = 0$$

yielding the general solutions as

$$Z(z) = Ae^{\lambda z} + Be^{-\lambda z} \quad \text{and} \quad \Theta(\theta) = C\cos\mu\theta + D\sin\mu\theta$$

Now, equation (2) reduces to

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\lambda^2 - \frac{\mu^2}{r^2}\right)R = 0$$

Which is Bessel's modified equation, having the solution

$$R(r) = EJ_\mu(\lambda r) + FJ_{-\mu}(\lambda r) \text{ for fractional } \mu$$

and $R(r) = EJ_\mu(\lambda r) + FY_\mu(\lambda r)$ for integral μ.

Hence, the combined solution is

$$V(r,\theta,z) = \sum_{\lambda,\mu}\left(Ae^{\lambda z} + Be^{-\lambda z}\right)\left(C\cos\mu\theta + D\sin\mu\theta\right)\left(EJ_\mu(\lambda r) + FJ_{-\mu}(\lambda r)\right),$$

for fractional μ.

$$V(r,\theta,z) = \sum_{\lambda,\mu}\left(Ae^{\lambda z} + Be^{-\lambda z}\right)\left(C\cos\mu\theta + D\sin\mu\theta\right)\left(EJ_\mu(\lambda r) + FY_\mu(\lambda r)\right),$$

for integral μ.

Corollary: 1. Taking constant $A_{\lambda\mu}$ and $B_{\lambda\mu}$, the general solution can be written as

$$R(r) = A_{\lambda\mu}J_\mu(\lambda r) + B_{\lambda\mu}Y_\mu(\lambda r)$$

But $Y_\mu(\lambda r) \to \infty$ as $r \to 0$, therefore if it is finite along the line $r = 0$, then $B_{\lambda\mu} = 0$, hence the solution is

$$V(r,\theta,z) = \sum_\lambda\sum_\mu A_{\lambda\mu}J_\mu(\lambda r)e^{\pm\lambda z \pm i\mu\theta}$$

Trying the superposition, we can write the solution as:

$$V(r,\theta,z) = \sum_{\lambda,\mu=0}^{\infty} J_\mu(\lambda r)\left[e^{\lambda z}(A_\mu\cos\mu\theta + B_\mu\sin\mu\theta) + e^{-\lambda z}(C_\mu\cos\mu\theta + D_\mu\sin\mu\theta\right]$$

2. **Solution of Laplace Equation in Two Dimension in Polar Co-ordinates**

The Laplace equation has the form:

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} = 0 \qquad ...(1)$$

To solve it by the method of separation of variables, we take

$$V\left(r,\theta\right) = R(r)\Theta(\theta) \qquad ...(2)$$

giving

$$\frac{\partial V}{\partial r} = \frac{dR}{dr}\ \Theta(\theta) \quad ; \quad \frac{\partial^2 V}{\partial r^2} = \frac{d^2 R}{dr^2}\Theta(\theta) \quad \text{and}$$

$$\frac{\partial^2 V}{\partial \theta^2} = R(r)\frac{d^2\Theta}{d\theta^2}$$

Substituting all these in the equation (1), we get

$$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr}\right) = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = n^2 (\text{say})$$

so that we have

$$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr}\right) = n^2 (\text{say})$$

$$\Rightarrow\ r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2 R = 0 \qquad \text{...(3)}$$

which is homogeneous and hence on putting

$$r = e^z, \quad \text{so that} \quad z = \log r \ \text{ and } \quad D = r\frac{d}{dr} = \frac{d}{dz}$$

then the equation (3) reduces to

$$\left[D(D-1) + D - n^2\right]r = 0$$

$$\Rightarrow \left(D^2 - n^2\right)r = 0$$

Its auxiliary equation is

$$D^2 - n^2 = 0$$

$$\Rightarrow D = \pm\ n$$

$$\therefore R(r) = Ae^{nz} + Be^{-nz}$$

$$= Ar^n + Br^{-n}$$

Also, the equation for (1) is

$$-\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = n^2 \qquad \text{...(4)}$$

It has the solution

$$\Theta(\theta) = C\cos n\theta + D\sin n\theta$$

The combined solution is

$$V_n(r,\theta) = \left(Ar^n + Br^{-n}\right)\left(C\cos n\theta + D\sin n\theta\right) \qquad \text{...(5)}$$

Also, for n=0, (3) and (4) becomes

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} = 0 \qquad ...(6)$$

and $$\frac{d^2\Theta}{d\theta^2} = 0 \qquad ...(7)$$

Having the solution of (6) and (7) as

$$R(r) = c_1 + c_2 \log r$$
$$\Theta(\theta) = d_1 + d_2\theta$$

Thus, for n=0, the solution is

$$V(r,\theta) = [c_1 + c_2 \log r][d_1 + d_2\theta]$$

Thus, the general solution is

$$V(r,\theta) = [c_1 + c_2 \log r][d_1 + d_2\theta] + \sum_{n=1}^{\infty}\left[A_n r^n + B_n r^{-n}\right][C_n \cos n\theta + D_n \sin n\theta]$$

1.5.5 Solution of Laplace Equation in Spherical Co-ordinates

In spherical polar co-ordinates, it has the form

$$r^2\frac{\partial^2 V}{\partial r^2} + 2r\frac{\partial V}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \qquad ...(1)$$

Assuming $V(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$, equation (1) becomes

$$\left[\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right] = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \lambda^2$$

$$\Rightarrow -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \lambda^2 \quad \Rightarrow \quad \frac{d^2\Phi}{d\phi^2} + \lambda^2\Phi = 0$$

with solution given by

$$\Theta(\phi) = Ce^{\pm i\lambda\phi}$$

and

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{\lambda^2}{\sin^2\theta} = n(n+1) \text{ (say)}$$

giving

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0 \qquad ...(2)$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[n(n+1) - \frac{\lambda^2}{\sin^2\theta}\right]\Theta = 0 \qquad ...(3)$$

Here (2), being homogeneous, if we put $r = e^s$ and $D \equiv \dfrac{d}{ds}$, reduces to

$$\left[D(D-1) + 2D - n(n+1)\right]R = 0$$

$$\text{or} \quad (D-n)(D+(n+1))R = 0$$

$$\therefore \quad R = Ae^{ns} + Be^{-(n+1)s}$$

$$= Ar^n + Br^{-n-1}$$

Putting $\mu = \cos\theta$ in (4), so that

$$\frac{d\Theta}{d\theta} = \frac{d\Theta}{d\mu}\frac{d\mu}{d\theta} = -\sin\theta\frac{d\Theta}{d\mu} \Rightarrow \frac{1}{\sin\theta}\frac{d}{d\theta} = -\frac{d}{d\mu}$$

we have

$$\frac{d}{d\mu}\left\{\left(1-\mu^2\right)\frac{d\Theta}{d\mu}\right\} + \left\{n(n+1) - \frac{\lambda^2}{1-\mu^2}\right\}\Theta = 0$$

$$\text{or } (1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + \left\{n(n+1) - \frac{\lambda^2}{1-\mu^2}\right\}\Theta = 0$$

which is associated Legendre equation, then the solution is of the form

$$\Theta = \Theta(\cos\theta)$$

and hence solution of given problem is

$$V_n(r,\theta,\phi) = (Ar^n + Br^{-n-1})\Theta(\cos\theta)e^{\pm i\lambda\phi}$$

Hence, summing overall n and trying superposition, the general solution of (1) may be expressed as

$$V(r,\theta,\phi) = \sum_{n,\lambda}(A_n r^n + \frac{B_n}{r^{n+1}})\Theta(\cos\theta)e^{\pm i\lambda\phi} \quad \text{is required solution.}$$

1.5.7 Solution of Three-Dimensional Wave Equation in Cartesian Co-ordinates

A partial differential equation of the form

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u$$

is known as Wave equation, that is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad ...(1)$$

with the conditions

$$\frac{\partial u}{\partial x} = 0 \text{ at } x = 0, x = a$$
$$\frac{\partial u}{\partial y} = 0 \text{ at } \mathrm{y} = 0, y = a$$
$$\frac{\partial u}{\partial z} = 0 \text{ at } \mathrm{z} = 0, z = a$$

and $u(x, y, z, t) \neq 0$ at $t = 0$

To solve the problem, we shall use the method of separation of variables and assume that

$$u(x, y, z, t) = X(x)Y(y)Z(z)T(t)$$

Now proceed as in previous examples to get

$$u_{n_1 n_2 n_3}(x, y, z, t) = \alpha_{n_1 n_2 n_3} \cos\frac{n_1\pi}{a} \cos\frac{n_2\pi}{a} \cos\frac{n_3\pi}{a} \cos\left(\frac{\pi ct}{a}\sqrt{n_1^2 + n_2^2 + n_3^2}\right)$$

Therefore, using the superposition, the general solution is

$$u(x, y, z, t) = \sum_{n_1, n_2, n_3 = 1}^{\infty} \alpha_{n_1 n_2 n_3} \cos\frac{n_1\pi}{a} x \cos\frac{n_2\pi}{a} y \cos\frac{n_3\pi}{a} z \cos\left(\frac{\pi ct}{a}\sqrt{n_1^2 + n_2^2 + n_3^2}\right)$$

Corollary: Wave equation in two-dimensional is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad ...(1)$$

And the solution is given by

$$u(x, y, t) = \sum_{n_1, n_2 = 1}^{\infty} \alpha_{n_1 n_2} \cos\frac{n_1\pi}{a} x \cos\frac{n_2\pi}{a} y \cos\left(\frac{\pi ct}{a}\sqrt{n_1^2 + n_2^2}\right)$$

1.5.8 Solution of three-dimensional Wave equation in cylindrical co-ordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad ...(1)$$

Let the solution is $u(r, \theta, z, t) = R(x)\Theta(\theta)Z(z)T(t)$ ...(2)

Choosing the constant the separation of variable such that

$$\frac{1}{c^2 T}\frac{d^2 T}{dt^2} = -\mathrm{p}^2 \quad \Rightarrow \frac{d^2 T}{dt^2} + \mathrm{p}^2 c^2 T = 0 \qquad ...(3)$$
$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -q^2 \quad \Rightarrow \frac{d^2\Theta}{d\theta^2} + q^2\Theta = 0 \qquad ...(4)$$
$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = -s^2 \quad \Rightarrow \frac{d^2 Z}{dz^2} + s^2 Z = 0 \qquad ...(5)$$

The equation (1) becomes

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{q^2}{r^2}-s^2=-p^2$$

$$\left(\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}\right)+\left(\xi^2-\frac{q^2}{r^2}\right)R=0 \quad ...(6)$$

where $\xi^2=-s^2+p^2$. Equation (6) is the modified Bessel's equation of order q has a solution

$$R(r)=A_3J_q(\xi r)+B_3J_{-q}(\xi r) \text{ for fractional } q$$

and $R(r)=A_3J_q(\xi r)+B_3Y_q(\xi r)$ for integral q.

Now

For a bounded solution, $Y_q(\xi r)\to\infty$ as $r\to 0$, therefore if it is finite along the line $r=0$, then $B_3=0$. Thus, the general solution of equation(1) is

$$u(r,\theta,z,t)=\sum_{p,q,s}CJ_q(\xi r)\left[A_1\cos(pct)+B_1\sin(pct)\right]\left[A_2\cos(q\theta)+B_2\sin(q\theta)\right]\left[A_3\cos(sz)+B_3\sin(sz)\right]$$

1.5.9 Solution of Three-dimensional Wave equation in Spherical co-ordinates

In polar spherical co-ordinates the Wave equation is

$$\frac{\partial^2u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2u}{\partial\phi^2}=\frac{1}{c^2}\frac{\partial^2u}{\partial t^2}$$

Assuming that the solution of (1) is

$$u(r,\theta,\phi,t)=R(r)\Theta(\theta)\Phi(\phi)T(t)$$

Now proceed as in previous articles to get

$$u(r,\theta,\phi,t)=\sum_{p,q,s}\left(A_1e^{\pm im\phi}\right)\left(A_2e^{\pm ipct}\right)\left(CP_n^m(\cos\theta)+DP_n^{-m}(\cos\theta)\right)\left(Er^{-\frac{1}{2}}J_{n+\frac{1}{2}}(pr)+Fr^{-\frac{1}{2}}J_{-\left(n+\frac{1}{2}\right)}(pr)\right) \quad \mathbf{1.6}$$

Method of separation of variables to solve B.V.P. associated with motion of a vibrating string

1.6.1 Solution of the problem of vibrating string with zero initial velocity and with initial displacement

Let us consider an elastic string of length L, fastened at its ends on the x-axis and assume that it vibrates in the $xy-$plane. Initially the string is released from the rest and we want to find out the expression for displacement function $y(x,t)$. The B.V.P. modeling the motion of string is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad (0 < x < L,\ t > 0) \qquad ...(1)$$

$$y(0,t) = y(L,t) = 0 \ \ (t > 0) \qquad ...(2)$$

$$y(x,0) = f(x), \qquad (0 < x < L) \qquad ...(3)$$

$$\frac{\partial y}{\partial t}(x,0) = 0, \qquad (0 < x < L) \qquad ...(4)$$

Here, it is assumed that $f(x)$ is the initial displacement of the string before release and initial velocity is zero. We find the solution of equation (1) by separation of variables. For this, we set $y(x,t) = X(x)T(t)$ and using this, we get

$$XT'' = a^2 X''T \quad or \quad \frac{X''}{X} = \frac{T''}{a^2 T}$$

Since the left side of this equation is a function of x only and right hand side is a function of t only where x and t are independent, both must equal to some constant. Let the constant of separation is $-\lambda$. The above equation has become

$$X'' + \lambda X = 0; \qquad T'' + \lambda a^2 T = 0 \quad ...(5)$$

Since $y(0,t) = X(0)T(t) = 0$

From here we conclude that $X(0) = 0$. This assumes that $T(t)$ is non-zero for some t. Otherwise $T = 0$ is zero for all time and we get the trivial solution, i.e., string would not move and it is possible only when $f(x) = 0$ means string is not displaced.

Similarly $y(L,t) = X(L)T(t) = 0$. Implies that $X(L) = 0$.

The problem for X is

$$X'' + \lambda X = 0$$
$$X(0) = 0 = X(L)$$

We have solved such types of problems earlier and the solution is

$$X_n(x) = A_n \sin\left(\frac{n\pi x}{L}\right) \text{ with eigen values } \lambda = \frac{n^2\pi^2}{L^2} \text{ for } n = 1, 2, 3...,$$

Now the problem of $T(t)$ is

$$T'' + \frac{n^2\pi^2}{L^2} T = 0$$

With the condition $\frac{\partial y(x,0)}{\partial t} = 0 \Rightarrow X(x)T'(0) = 0 \ \Rightarrow T'(0) = 0.$ Otherwise the solution becomes trivial.

Thus the general solution is

$$T_n(t) = C_n \cos\left(\frac{n\pi at}{L}\right) + D_n \sin\left(\frac{n\pi at}{L}\right)$$

By applying the condition $T'(0) = 0$, we get $D_n = 0$. Hence for fixed n, the solution for

$T_n(t) = C_n \cos\left(\frac{n\pi at}{L}\right)$ for $n = 1, 2, 3...$

Now, for a fixed n, the solution of equation (1) is

$$y_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi at}{L}\right) \quad where\ B_n = A_n C_n$$

Using the superposition, we obtain

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi at}{L}\right)$$

Now, using the condition (3), we have

$$y(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Which is Fourier sine series and the value of constant coefficient B_n is $\frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi$

Thus, we have

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right]\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi at}{L}\right)$$

Corollary: In the above problem, if $f(x)$ is replaced by

$$f(x) = \begin{cases} x & , 0 \le x \le \frac{L}{2} \\ L - x & , \frac{L}{2} \le x \le L \end{cases}$$

The coefficient

$$\begin{aligned} B_n &= \frac{2}{L}\left[\int_0^{\frac{L}{2}} \xi \sin\left(\frac{n\pi\xi}{L}\right)d\xi + \int_{\frac{L}{2}}^{L} (L-\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right] \\ &= \frac{4L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

Thus the solution becomes

$$y(x,t)=\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi at}{L}\right)$$

Since $\sin\left(\frac{n\pi}{2}\right)=0$ if n is even and $\sin\left(\left(2k-1\right)\frac{\pi}{2}\right)=\left(-1\right)^{k-1}$ if n is odd positive integer. The solution is

$$y(x,t)=\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(-1\right)^{k-1}}{\left(2n-1\right)^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi at}{L}\right)$$

1.6.2 Solution of the Problem of Vibrating String with Initial Velocity and Zero Initial Displacement

The B.V.P is

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}\qquad \left(0<x<L,\ t>0\right)\qquad ...(1)$$

$$y\left(0,t\right)=y\left(L,t\right)=0\qquad \left(t>0\right)\qquad ...(2)$$

$$y\left(x,0\right)=0,\qquad (0<x<L)\qquad ...(3)$$

$$\frac{\partial y}{\partial t}(x,0)=g(x),\qquad \left(0<x<L\right)\qquad ...(4)$$

Similarly to earlier article, the solution for X is

$$X_n(x)=A_n\sin\left(\frac{n\pi x}{L}\right)\text{ with eigen values }\lambda=\frac{n^2\pi^2}{L^2}\text{ for }n=1,2,3...,$$

The solution for T is

$$T_n(t)=C_n\cos\left(\frac{n\pi at}{L}\right)+D_n\sin\left(\frac{n\pi at}{L}\right)$$

And applying the condition (3), we have

$$y(x,0)=X(x)T(0)=0\Rightarrow T(0)=0$$

This implies $C_n=0$ and the solution for T is $T_n(t)=\sum_{n=1}^{\infty}D_n\sin\left(\frac{n\pi at}{L}\right)$

Therefore, the general solution is

$$y(x,t)=\sum_{n=1}^{\infty}y_n(x,t)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi at}{L}\right)\quad where\ A_nD_n=B_n\ \ldots\ (5)$$

Now, using condition (4), we have

$$\frac{\partial y}{\partial t}(x,0) = g(x)$$
$$\frac{\partial y}{\partial t}(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

Which is a Fourier sine series for $g(x)$, the value of coefficient B_n is

$$B_n = \frac{2}{n\pi a}\left[\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right]$$

$$\therefore \quad y(x,t) = \frac{2}{n\pi a}\sum_{n=1}^{\infty}\left[\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right]\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi at}{L}\right)$$

Example: Solve the following B.V.P

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \qquad (0 < x < L,\ t > 0) \qquad ...(1)$$
$$y(0,t) = y(L,t) = 0 \quad (t > 0) \qquad ...(2)$$
$$y(x,0) = 0, \qquad (0 < x < L) \qquad ...(3)$$
$$\frac{\partial y}{\partial t}(x,0) = \begin{cases} x & ,0 \le x \le \frac{L}{4} \\ 0 & ,\frac{L}{4} \le x \le L \end{cases} \qquad ...(4)$$

1.6.3 The solution of the String Problem with Initial Velocity and with Displacement

Consider a string with both initial displacement $f(x)$ and initial velocity $g(x)$. To solve this problem, we firstly, formulate two separate problems, the first with initial displacement $f(x)$ and zero initial velocity, and the second with zero initial displacement and initial velocity $g(x)$. In earlier article, we solved the problem of string with zero initial velocity and with displacement and initial velocity and with zero displacement. Let $y_1(x,t)$ be the solution of the first problem, and $y_2(x,t)$ the solution of the second. Now let $y(x,t) = y_1(x,t) + y_2(x,t)$. Then y satisfies the Wave equation and the boundary conditions.

1.7 Solution of Wave equation for Semi-infinite and Infinite Strings

1.7.1 Wave Motion for a Semi-infinite String

Let us consider an elastic string which is fixed at $x = 0$ and stretched from 0 to ∞. The B.V.P. for the motion of semi-infinite string is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad (x > 0,\ t > 0) \qquad ...(1)$$

$$y(0,t) = 0 \qquad (t > 0) \qquad ...(2)$$

$$y(x,0) = f(x), \qquad (x > 0) \qquad ...(3)$$

$$\frac{\partial y}{\partial t}(x,0) = g(x), \qquad (x > 0) \qquad ...(4)$$

Here, this the problem of vibrating string with initial velocity and displacement. So we will separate the problem in two parts: (i) zero initial velocity and with displacement (ii) with initial velocity and zero displacement.

(i) Zero initial velocity

For this case $g(x) = 0$. For a bounded solution, we firstly set $y(x,t) = X(x)T(t)$.

Using this in equation, we get

$$\frac{X''}{X} = \frac{T''}{a^2 T} \qquad \dots (5)$$

In equation (5), the left side is a function of x only while right side is function of t. So each side must be equal to some constant, let that separation of constant is λ. The equation (5) becomes

$$\begin{aligned} X'' + \lambda X &= 0 \\ T'' + \lambda a^2 T &= 0 \end{aligned} \qquad \dots (6)$$

And the condition (2) and (4) becomes

$$y(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0 \qquad ...(7)$$

$$\frac{\partial y}{\partial t}(x,0) = X(x)T'(0) = 0 \Rightarrow T'(0) = 0 \quad ...(8)$$

Now we will discuss the cases for different values of λ.

Case 1: If $\lambda = 0$

Then $X'' = 0 \quad \Rightarrow X(x) = Ax$

Which is unbounded solution on the given domain, unless $A = 0$. Thus, for this case we have a trivial solution.

Case 2: if $\lambda < 0$, let $\lambda = -p^2$ with $p > 0$.

Then $X'' - p^2 X = 0$ with the solution

$$X(x) = Ae^{px} + Be^{-px}$$

Now,
$$X(0) = 0 \Rightarrow A + B = 0 \Rightarrow A = -B$$
$$\therefore \quad X(x) = A\left(e^{px} - e^{-px}\right)$$

which is unbounded solution for $p>0$ unless $A=0$. Thus $X(x)=0$, again we get a trivial solution.

Case 3: if $\lambda>0$, let $\lambda=p^2$ with $p>0$.

Then $X''+p^2X=0$ with the solution

$$X(x)=A\cos px+B\sin px$$

Now,
$$X(0)=0 \Rightarrow C=0$$
$$\therefore \quad X(x)=D\sin px$$

Thus for each $p>0$, $\lambda=p^2$ is an eigen value and $X_p(x)=D_p\sin px$.

Now the problem for T is

$$T''+a^2p^2T=0 \text{ with the solution}$$
$$T(t)=C\cos(pat)+D\sin(pat)$$

Now,
$$T'(0)=0 \Rightarrow paD=0 \Rightarrow D=0$$
$$T(t)=C\cos(pat)$$

Thus for $p>0, \quad T_p(t)=C_p\cos\left(pat\right)$

Therefore, for this case

$$y_p(x,t)=E_p\sin\left(px\right)\cos\left(pat\right) \quad where\ E_p=A_pC_p$$

Using the superposition, we have

$$y(x,t)=\int_0^\infty E_p\sin\left(px\right)\cos\left(pat\right)dp$$

Also it is given $y(x,0)=f(x)\Rightarrow\int_0^\infty E_p\sin\left(px\right)dp=f(x)$

So
$$E_p=\frac{2}{\pi}\int_0^\infty f(\xi)\sin\left(p\xi\right)d\xi$$

(i) Zero initial displacement

For this case $f(x)=0$. For a bounded solution, we firstly set $y(x,t)=X(x)T(t)$.

Using this in equation, we get

$$\frac{X''}{X}=\frac{T''}{a^2T}$$

In equation, the left side is a function of x only while right side is function of t. So each side must be equal to some constant, let that separation of constant is λ. The equation becomes

$$X''+\lambda X=0$$
$$T''+\lambda a^2 T=0$$

And the condition (2) and (4) becomes

$$y(0,t)=X(0)T(t)=0 \Rightarrow X(0)=0$$
$$y(x,0)=X(x)T(0)=0 \Rightarrow T(0)=0$$

Now we will discuss the cases for different values of λ.

Case 1: If $\lambda=0$

Then $X''=0 \quad \Rightarrow X(x)=Ax$

Which is unbounded solution on the given domain, unless $A=0$. Thus, for this case we have a trivial solution.

Case 2: if $\lambda<0$, let $\lambda=-p^2$ with $p>0$.

Then $X''-p^2X=0$ with the solution

$$X(x)=Ae^{px}+Be^{-px}$$

Now,
$$X(0)=0 \Rightarrow A+B=0 \Rightarrow A=-B$$
$$\therefore \quad X(x)=A\left(e^{px}-e^{-px}\right)$$

Which is unbounded solution for $p>0$ unless $A=0$. Thus $X(x)=0$, again we get a trivial solution.

Case 3: if $\lambda>0$, let $\lambda=p^2$ with $p>0$.

Then $X''+p^2X=0$ with the solution

$$X(x)=A\cos px+B\sin px$$

Now,
$$X(0)=0 \Rightarrow C=0$$
$$\therefore \quad X(x)=D\sin px$$

Thus for each $p>0$, $\lambda=p^2$ is an eigen value and $X_p(x)=D_p\sin px$.

Now the problem for T is

$$T''+a^2p^2T=0 \text{ with the solution}$$
$$.T(t)=C\cos(pat)+D\sin(pat).$$

Now,
$$T(0)=0 \Rightarrow paC=0 \Rightarrow C=0$$
$$T(t)=D\sin(pat)$$

Thus for $p>0, \quad T_p(t)=D_p\sin\left(pat\right)$

Therefore, for this case

$$y_p(x,t) = E_p \sin(px)\sin(pat) \quad where\ E_p = A_p D_p$$

Using the superposition, we have

$$y(x,t) = \int_0^\infty E_p \sin(px)\sin(pat)\,dp$$

Also it is given

$$\frac{\partial y(x,0)}{\partial t} = g(x) \Rightarrow \int_0^\infty paE_p \sin(px)\,dp = g(x)$$

$$E_p = \frac{2}{pa\pi}\int_0^\infty f(\xi)\sin(p\xi)\,d\xi$$

Thus, the general solution is

$$y(x,t) = \frac{2}{a\pi}\int_0^\infty \left[\frac{1}{p}\int_0^\infty f(\xi)\sin(p\xi)\,d\xi\right]\sin(px)\cos(pat)\,dp$$

1.7.2 Wave Motion for a Infinite String

Let us consider an elastic string which stretched over a real line. The B.V.P. for the motion of infinite string is

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \qquad (-\infty < x < \infty,\ t > 0) \qquad ...(1)$$

$$y(x,0) = f(x), \qquad (-\infty < x < \infty) \qquad ...(2)$$

$$\frac{\partial y(x,0)}{\partial t} = g(x), \qquad (-\infty < x < \infty) \qquad ...(3)$$

Similar to previous article, we will separate the problem in two parts: (i) zero initial velocity and with displacement (ii) with initial velocity and zero displacement.

Case (i) Zero initial velocity

For this case $g(x) = 0$. For a bounded solution, we firstly set $y(x,t) = X(x)T(t)$. Using this in equation, we get

$$\frac{X''}{X} = \frac{T''}{a^2T}$$

In equation, the left side is a function of x only while right side is function of t. So each side must be equal to some constant, let that separation of constant is λ. The equation becomes

$$X'' + \lambda X = 0$$

$$T'' + \lambda a^2 T = 0$$

and the condition becomes

$$\frac{\partial y}{\partial t}(x,0) = X(x)T'(0) = 0 \;\Rightarrow T'(0) = 0$$

Now we will discuss the cases for different values of λ .

Case 1: If $\lambda = 0$

Then $X'' = 0 \quad \Rightarrow X(x) = Ax + B$

Which is unbounded solution on the given domain, unless $A = 0$. Thus solution is $X(x) = B$ for the eigen value.

Case 2: if $\lambda < 0$, let $\lambda = -p^2$ with $p > 0$.

Then $X'' - p^2 X = 0$ with the solution

$$X(x) = Ae^{px} + Be^{-px}$$

Since $p > 0$, the first term in right hand side Ae^{px} is unbounded in the domain $[0, \infty)$ and the second term Be^{-px} is unbounded in the region $(-\infty, 0)$ Therefore, for a bounded solution, we have to assume that $A = 0 \;\; and \;\; B = 0$. Therefore $X(x) = 0$

Case 3: if $\lambda > 0$, let $\lambda = p^2$ with $p > 0$.

Then $X'' + p^2 X = 0$ with the solution

$$X(x) = A\cos px + B\sin px$$

The function $X(x)$ is always bounded for every $p > 0$ an so, we have

$$X_p(x) = A_p\cos(px) + B_p\sin(px)$$

Now the problem for T is

$T'' + \lambda a^2 T = 0$ and

$$\frac{\partial y}{\partial t}(x,0) = X(x)T'(0) = 0 \;\Rightarrow T'(0) = 0$$

If $\lambda = 0,$ then we have
$$\begin{aligned} &T(t) = Ct + D \\ &and \quad T'(0) = 0 \;\Rightarrow C = 0 \\ &\therefore \; T(t) = D \end{aligned}$$

Is a solution for T . On the other hand, if $\lambda = p^2, \;\; p > 0$, then the equation $T'' + a^2 p^2 T = 0$ has the solution

$$T(t) = E\cos(pat) + F\sin(pat)$$

But $T'(0)=0 \Rightarrow paF=0 \Rightarrow F=0$

$\therefore \quad T_p(t)=E_p\cos(pat)$

Therefore, for this case

$$y_p(x,t)=\left[a_p\cos(px)+b_p\sin(px)\right]\cos(pat) \quad where\ a_p=A_pE_p \quad and \quad b_p=B_pE_p$$

Using the superposition, we have

$$y(x,t)=\int_{-\infty}^{\infty}\left[a_p\cos(px)+b_p\sin(px)\right]\cos(pat)\,dp$$

Also it is given $y(x,0)=f(x)=\int_{-\infty}^{\infty}\left[a_p\cos(px)+b_p\sin(px)\right]\cos(pat)\,dp$

Where

$$a_p=\frac{1}{\pi}\int_{-\infty}^{\infty}f(\xi)\cos(p\xi)\,dp$$

$$b_{p=}\frac{1}{\pi}\int_{-\infty}^{\infty}f(\xi)\sin(p\xi)\,dp$$

So $E_p=\frac{2}{\pi}\int_0^{\infty}f(\xi)\sin(p\xi)\,d\xi$

(ii) **Zero initial displacement**

For this case $f(x)=0$. Similar to previous case, the eigen function for X is

$X_p(x)=A_p\cos(px)+A_p\sin(px)$ with eigen values $\lambda=p^2$ with $p>0$. And the solution for T

$$T_p(t)=E_p\cos(pat)+F_p\sin(pat)$$

The problem is same as zero initial velocity except the condition $y(x,0)=0$.This implies $X(x)T(0)=0 \Rightarrow T(0)=0$. We have $E_p=0$. The solution becomes $T_p(t)=F_p\sin(pat)$

Therefore, for this case, the solution is

$$y_p(x,t)=\left[a_p\cos(px)+b_p\sin(px)\right]\sin(pat) \quad where\ a_p=A_pF_p \ \ and\ \ b_p=B_pF_p$$

Using the superposition, we have

$$y(x,t)=\int_{-\infty}^{\infty}\left[a_p\cos(px)+b_p\sin(px)\right]\sin(pat)\,dp$$

Also it is given $\frac{\partial y(x,0)}{\partial t}=g(x) \Rightarrow \int_{-\infty}^{\infty}pa\left[a_p\cos(px)+b_p\sin(px)\right]\sin(pat)\,dp=g(x)$.

The coefficient a_p and b_p are given by

$$a_p = \frac{1}{ap\pi}\int_{-\infty}^{\infty} g(\xi)\cos(p\xi)\,d\xi$$

$$b_p = \frac{1}{ap\pi}\int_{-\infty}^{\infty} g(\xi)\sin(p\xi)\,d\xi$$

Problems:

1. Find the solution of B.V.P

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \qquad (0 < x < L,\ t > 0)$$

$$y(0,t) = y(L,t) = 0 \quad (t > 0)$$

$$y(x,0) = \begin{cases} x & ,0 \le x \le \dfrac{L}{2} \\ L - x & ,\dfrac{L}{2} \le x \le L \end{cases}$$

$$\frac{\partial y}{\partial t}(x,0) = x\left(\cos\left(\frac{\pi x}{L}\right)\right) \qquad (0 < x < L)$$

2. Find the solution of B.V.P.

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \qquad (x > 0,\ t > 0)$$

$$y(0,t) = 0 \qquad (t > 0)$$

$$y(x,0) = \begin{cases} x(1-x) & ,0 < x < 1 \\ 0 & ,x > 1 \end{cases}$$

$$\frac{\partial y}{\partial t}(x,0) = 0 \qquad (x > 0)$$

Some other problems

The Heat Equation in an Infinite Cylinder

Suppose we want the temperature distribution in a solid, infinitely long, homogeneous circular cylinder of radius R. Let the z-axis be along the axis of the cylinder. In cylindrical co-ordinates the Heat equation is:

$$\frac{\partial u}{\partial t} = a^2\nabla^2 u = a^2\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

We assume that the temperature at any point in the cylinder depends only on the time t and the distance r from the z-axis, the axis of the cylinder. This means that $\dfrac{\partial u}{\partial \theta} = \dfrac{\partial u}{\partial z} = 0$ and the Heat equation is

$$V_n(r,\theta,\phi)=(Ar^n+Br^{-n-1})\Theta(\cos\theta)e^{\pm i\lambda\phi}$$

$$\frac{\partial u}{\partial t}=a^2\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right]$$

Here u is a function of r and t only. The boundary condition we will consider is

$$u(R,t)=0$$

for $t>0$. This means that the outer surface is kept at temperature zero.

Now as in previous articles (left as an exercise for readers) we obtain,

$$u(r,t)=\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_0\left(\frac{z_n r}{R}\right)}{J_1^2(z_n)}e^{\frac{-a^2 z_n^2 t}{R^2}}\int_0^R f(\xi)\xi J_0\left(\frac{z_n\xi}{R}\right)d\xi$$

Solve the following exercise:

Exercise: A homogeneous circular cylinder of radius 2 and semi-infinite length has its base, which is sitting on the plane $z=0$, maintained at a constant positive temperature K. The lateral surface is kept at temperature zero. Determine the steady-state temperature of the cylinder if it has a thermal diffusivity of a^2, assuming that the temperature at any point depends only on the height z above the base and the distance μ from the axis of the cylinder.

The Heat Equation in a Solid Sphere:

Consider a solid sphere of radius R centered at the origin. We want to solve for the steady-state temperature distribution, given the temperature at all times on the surface

Solution: Here, it is natural to use spherical co-ordinates. We assume that temperature depends only on distance from the origin R. The angle of declination from the z-axis ϕ, with $\frac{\partial u}{\partial\theta}=0$, Laplace equation in spherical co-ordinates is

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta}=0 \qquad ...(1)$$

for $(0<r\le R, 0\le\phi\le\pi)$

The temperature is given on the surface

$$u(R,\phi)=f(\phi)\quad(0\le\phi\le\pi) \qquad ...(2)$$

To solve this BVP, we set

$$u(r,\phi)=R(r)\Theta(\phi) \qquad ...(3)$$

(Remaining solution is left for readers as an exercise).

CHAPTER-2

LAPLACE EQUATION AND ITS SOLUTION

Structure

2.1 Introduction

To model the physical problems, the partial differential equations (PDEs) are the common method**.** PDEs can be used to describe a wide variety of phenomena such as sound, heat, diffusion, electrostatics, electrodynamics, fluid dynamics, elasticity, gravitation and quantum mechanics, etc. In this chapter, we will discuss about different types of the partial differential equations, their classifications and the classical and weak solutions, etc.

Partial Differential Equation

A partial differential equation (PDE) is differential equation that contain an unknown function and its partial derivate with respect to two or more variables i.e., let U be an open subset of R^n. An expression of the form

$$F(D^k u(x), D^{k-1}u(x), ..., Du(x), u(x), x) = 0 \quad (x \in U) \qquad \ldots(1)$$

is called a kth-order partial differential equation, where

$F : R^{n^k} \times R^{n^{k-1}} \times ... \times R^n \times R \times U \to R$ is given and $u : U \to R$ is the unknown.

Example: The equation $u_t + u_x = 0$ is a partial differential equation, the unknown function is u and independent variables are x and t.

2.1.1 Classifications of Partial Differential Equations

Partial Differential Equations can be classified into four types

(a) Linear (b) Semi-linear (c) Quasi-linear (d) Non-linear.

(a) **Linear Partial Differential Equation**: A Partial Differential Equation (1) is said to linear PDE if it has the form

$$\sum_{|\alpha|\le k} a_\alpha(x)D^\alpha u = f(x) \qquad \dots (2)$$

for a given function $a_\alpha\left(\alpha\le k\right)$ and f Here, it is clear that the coefficients of derivate are a function of x only. The above equation is said to be homogeneous if *f=0.*

For example: $u_t + u_x = 0$ is a transport equation which is of first order, linear and homogeneous.

Some famous linear PDE are

1. Laplace equation $\Delta u = 0$ or $\sum_i u_{xx} = 0$
2. Linear Transport Equation $u_t + \bar{b}\Delta u = 0,\ \bar{b}\in R^n$
   $Du = \left(u_{x_1}, u_{x_2} ... u_{x_n}\right)$
3. Heat (Diffusion) Equation $u_t - \Delta u = 0$
4. Wave equation $u_{tt} - \Delta u = 0$

(b) **Semi-linear Partial Differential Equation**: A Partial Differential Equation (1) is said to semi-linear PDE if it has the form

$$\sum_{|\alpha|=k} a_\alpha(x)D^\alpha u + a_0\left(D^{\alpha-1}u,...,Du,u,x\right) = 0, \qquad \dots (3)$$

Here, coefficient of highest order derivative is a function of x only.

For example: $a(x)u_{xx} + u_x u_t = 0$.

(c) Quasi-linear Partial Differential Equation: A Partial Differential Equation (1) is said to quasi PDE if it has the form

$$\sum_{|\alpha|=k} a_\alpha(D^{\alpha-1}u,...,Du,u,x)D^\alpha u + a_0\left(D^{\alpha-1}u,...,Du,u,x\right) = 0, \qquad \dots (4)$$

Here, coefficient of highest order derivative are lower order derivative and function of *x* but not same order derivatives.

For example: $u_x u_{xx} + u_x u_t = 0$

(d) Nonlinear Partial Differential Equation: A Partial Differential Equation is non-linear in the highest order derivatives.

For example: $u^2 u_{xx} + u_x u_t = 0$

2.1.2 System of Partial Differential Equations

An expression of the form is said to be **system of partial differential equations** if it is represented by

$$\overline{F}(D^k\overline{u}(x), D^{k-1}\overline{u}(x), ...D\overline{u}(x), \overline{u}(x), x) = 0 \quad (x \in U)$$

is called a *k*th order system of partial differential equations in *u* where

$$\overline{F}: R^{mn^k} \times R^{mn^{k-1}} \times ... \times R^{mn} \times R^m \times U \to R^m$$

is given and $\overline{u} = \left(u^1, u^2, ..., u^m\right)$ is the unknown function such that $\overline{u}: U \to R^m$

For example:

$$\mu\Delta\overline{u} + (\lambda + \mu)\, div\overline{u} = 0 \quad \text{where } \overline{u} = \left(u^1, u^2, u^3\right)$$

Note: The classifications of system of partial differential equations are same as in case of a partial differential equations**.**

2.1.3 Solution of PDE

An expression u which satisfies the given PDE (1) is called a solution of the Partial Differential Equation.

Well posed problem: A given problem in Partial Differential Equation is well posed (Hadaward) if it satisfies

(i) existence
(ii) uniqueness
(iii) continuously depend on the data of given problem.

Classical Solution: If a solution of a given problem satisfies the above three conditions i.e., the solution of k^{th} order partial differential equation exists, is unique and is at least k times differentiable, then the solution is called classical solution. Solutions of Wave equation, Lalpace, and Heat equation etc., are classical solutions.

Weak Solution: If a solution of a given problem exists and is unique but does not satisfy the conditions of differentiability, such solution is called weak solution.

For Example: The gas conservation equation

$$u_t + F(u_x) = 0$$

models a shock wave in particular situation. So solutions exists, is unique, but not continuous. Such solution is known as weak solution.

Note: There are several physical phenomenon in which the problem has a unique solution, but does not satisfy the condition of differentiability. In such cases, we cannot claim that we are not able to find the solution rather such solutions are called weak solutions

2.2 Transport Equation

Homogeneous Transport Equation

The simplest partial differential equation out of four important equations is the Transport equation with constant coefficient

$$u_t + b\,.\,Du = 0 \qquad \dots (1)$$

in $R^n \times (0,\infty)$, where $b = (b_1, b_2, b_3 ..., b_n)$ is a fixed vector in R^n and $u : R^n \times [0,\infty) \to R$ is the unknown function $u=u(x, t)$. Here $x = (x_1, ..., x_n) \in R^n$ denotes a typical point in space and $t \geq 0$ is the time variable.

Theorem: Initial Value Problem

Consider the homogeneous transport equation

$$u_t + b\,.\,Du = 0 \quad \text{in} \quad R^n \times [0,\infty) \qquad \dots(1)$$

$$u = g \quad on \quad R^n \times \{t = 0\} \qquad \dots(2)$$

where $b \in R^n$ and $g : R^n \to R$ is known and $Du = D_x u = (u_{x_1}, ..., u_{x_n})$ for the gradient of u with respect to the spatial variables x. The problem is to compute $u(x,t)$.

Solution:

Let (x,t) be any point in the $R^n \times [0,\infty)$. To solve equation (1) , we observe the L.H.S. of equation (1) carefully, we find that it denotes the dot product of $(u_{x_1}, ..., u_{x_n}, u_1)$ with $(b_1, ..., b_n, 1)$. So L.H.S. of equation (1) tells that the derivative of u in the direction of $(b,1)$ is zero in R^{n+1} dimensional space. So, the line through (x,t) in the direction of $(b,1)$ is

$$\left.\begin{aligned} x(s) &= x + sb \\ t(s) &= t + s \end{aligned}\right\}, \quad s \in R \qquad \dots (3)$$

This line hits the plane $\Gamma := R^n \times \{t = 0\}$ at the point $(x - tb, 0)$ when $s = -t$.

Define a parametric equation of line in the direction $(b,1)$ is

$$z(s) = u(x + sb, t + s) \qquad \dots (4)$$

where $s \in R$ is the parameter and $z : R \to R$.

Then, differentiating (4) w.r.t. s, we get

$$\begin{aligned} \dot{z}(s) &= Du(x + sb, t + s).b + u_t(x + sb, t + s) \\ &= 0 \end{aligned} \qquad \text{(using (1))}$$

$\Rightarrow z(s)$ is a constant function of s on the line (3).

$\Rightarrow$ u is constant on the line (4) through (x,t) with the direction $(b,1) \in R^{n+1}$.

and $u(x-tb,0) = g(x-tb)$

By virtue of given initial condition (2), we deduce that

$u(x,t) = g(x-tb)$ … (5) for $x \in R^n$ and $t \geq 0$.

Hence, if we know the value of u at any point on each such line, we know its value everywhere in $R^n \times (0,\infty)$ and it is given by equation (5).

Conversely, if $g \in C^1$, then $u = u(x,t)$ defined by (5) is indeed a solution of given initial value problem.

From (5), we find that

$$u_t = -b.D(x-tb)$$

and

$$Du = Dg$$

Hence $u_t + b.Du = -b.Dg + b.Dg = 0$ for (x ,t) in $R^n \times [0,\infty)$ and for t=0 $u(x,0) = g(x)$ on $R^n \times \{t=0\}$

Remark: If g is not C^1, then there is obviously no C^1 solution of (1). But even in this case formula (5) certainly provides a strong and in fact the only reasonable, candidate for a solution. We may thus informally declare $u(x,t) = g(x-tb)(x \in R^n, t \geq 0)$ to be a weak solution of given initial value problem even should g not be C^1. This all makes sense even if g and thus u are discontinuous. Such a notion, that a non-smooth or even discontinuous function may sometimes solve a PDE will come up again later when we study nonlinear transport phenomenon.

2.3 Non-homogenous Problem

Theorem: Consider the non–homogeneous initial value problem of transport equation

$$u_t + b.Du = f(x,t) \quad in \quad R^n \times (0,\infty) \qquad ...(1)$$

$$u = g \qquad \text{on } R^n \times \{t=0\} \qquad ...(2)$$

where $b \in R^n$, $g : R^n \to R$, $f : R^n \times [0,\infty) \to R$ is known and $Du = D_x u = \left(u_{x_1}, ..., u_{x_n}\right)$ for the gradient of u with respect to the spatial variables x. Solve the equation for u=u(x,t) with initial condition (2).

Solution: Fix a point $(x,t) \in R^{n+1}$, as discussed before, the equation of line passing through (x,t) in the direction of $(b,1)$ is given by $z(s) = u(x+sb, t+s)$, where s is the parameter.

Differentiating this w. r. t. s

$$\dot{z}(s) = f(x+sb, t+s) \qquad \text{(using (1))}$$

Integrating w. r. t. s from –t to 0

$$\int_{-t}^{0} \dot{z}(s)ds = \int_{-t}^{0} f(x+sb, t+s)ds$$

$$z(0)-z(-t) = \int_{-t}^{0} f(x+sb, t+s)ds$$

Substitute t+s=ψ , ds=dψ

$$z(0)-z(-t) = \int_{0}^{t} f\left(x+b(\psi-t), \psi\right)d\psi$$

$$u(x,t)-u(x-bt,0) = \int_{0}^{t} f\left(x+b(s-t), s\right)ds \qquad (\because \text{replacing } \psi \text{ by } s)$$

$$u(x,t) = u(x-bt,0) + \int_{0}^{t} f\left(x+b(s-t), s\right)ds$$

$$u(x,t) = g(x-bt) + \int_{0}^{t} f\left(x+b(x-t), s\right)ds \qquad \left(x \in R^n, t \geq 0\right)$$

It is the required solution of initial value problem for non-homogeneous transport equation.

2.4 Laplace's Equation and its Fundamental Solution

We get the Laplace's equation in several physical phenomenon such as irrotational flow of incompressible fluid, diffusion problem etc. Let $U \subset R^n$ be a open set, $x \in R$ and the unknown is $u : \bar{U} \to R$, $u = u(x)$ then, the Laplace's equation is defined as

$$\Delta u = 0 \qquad \ldots (1)$$

and Poisson's equation

$$-\Delta u = f$$

where the function $f : U \to R$ is given.

and also remember that the Laplacian of u is $\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$.

Definition: Harmonic function

A C^2 function u satisfying the Laplace's equation $\Delta u = 0$ is called a harmonic function.

Theorem: Find the fundamental solution of the Laplace's equation (1).

Solution: Probably, it is to be noted that the Laplace equation is invariant under rotation. So we attempt to find a solution of Laplace's equation (1) in $U = R^n$, having the form (radial solution)

$$u(x) = v(r), \qquad \ldots(2)$$

where $r = |x| = \left(x_1^2 + ... + x_n^2\right)^{1/2}$ and v is to be selected (if possible) so that $\Delta u = 0$ holds.

We note that

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}\left(x_1^2 + ... + x_n^2\right)^{-1/2} 2x_i = \frac{x_i}{r} \qquad (x \neq 0)$$

for i=1,2,…,n.

Thus, we have

$$u_{x_i} = v'(r)\frac{x_i}{r},$$

and $u_{x_i x_i} = v''(r)\dfrac{x_i^2}{r^2} + v'(r)\left(\dfrac{1}{r} - \dfrac{x_i^2}{r^3}\right)$

for i=1,…,n.

So

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = \sum_{i=1}^{n} v''(r)\left(\frac{x_i^{\,2}}{r}\right) + v'(r)\left\{\sum_{i=1}^{n}\left(\frac{1}{r} - \frac{x_i^{\,2}}{r^2}\right)\right\} = v''(r) + \frac{n-1}{r}v'(r)$$

Hence $\Delta u = 0$ if and only if

$$v'' + \frac{n-1}{r}v' = 0$$

If $v' \neq 0$, we deduce

$$\log\left(|v'|\right)' = \frac{v''}{v'} = \frac{1-n}{r},$$

Integrating w. r. t. r,

$$\log v' = -(n-1)\log r + \log a$$

where $\log a$ is a constant.

Now, $\qquad v' = \dfrac{a}{r^{n-1}}$

Again integrating

$$v(r) = \begin{cases} a\log r + b & (n = 2) \\ \dfrac{a}{r^{n-2}} + b & (n \geq 3) \end{cases}$$

where a and b are constants.

Therefore, if $r > 0$, the solution of Laplace's equation is

$$u(x) = \begin{cases} a \log|x| + b & (n=2) \\ \dfrac{a}{|x|^{n-2}} + b & (n \geq 3) \end{cases}$$

Without loss of generality, we take b=0. To find b, we normalize the solution i.e.

$$\int_{R^n} u(x) dx = 1$$

So, the solution is

$$u(x) = \begin{cases} -\dfrac{1}{2\pi} \log|x| & (n=2) \\ \dfrac{1}{n(n-2)\alpha(n)|x|^{n-2}} & (n \geq 3) \end{cases} \qquad \ldots (3)$$

for each $x \in R^n, x \neq 0$ and $\alpha(n)$ is the volume of the unit ball in R^n.

We denote this solution by $\Phi(x)$ and

$$\Phi(x) = \begin{cases} \dfrac{-1}{2\pi} \log|x| & (n=2) \\ \dfrac{1}{n(n-2)\alpha(n)|x|^{n-2}} & (n \geq 3) \end{cases} \qquad \ldots (4)$$

defined for $x \in R^n, x \neq 0$, is the fundamental solution of Laplace's equation.

Remarks: 1. We conclude that the solution of Laplace's equation $\Delta u = 0, \Phi(x)$ is harmonic for $x \neq 0$. So the mapping $x \to \Phi(x)$, $x \neq 0$ is harmonic.

2. Shifting the origin to a new point y, the Laplace's equation remains unchanged. So $\Phi(x-y)$ is harmonic for $x \neq y$. If $f : R^n \to R$ is harmonic, then $\Phi(x-y) f(y)$ is harmonic for each $y \in R^n$ and $x \neq y$.

3. If we take the sum of all different points y over R^n, then

$$\int_{R^n} \Phi(x-y) f(y) dy \quad \text{is harmonic.}$$

Since $\Delta u(x) = \int_{R^n} \Delta_x \Phi(x-y) f(y) dy$

is not valid near the singularity $x = y$.

We must proceed more carefully in calculating Δu.

2.4.1 Fundamental Solution of Poission's Equation

To solve the Poission equation is $\Delta u = -f$, where $x \in U \subseteq R^n$, $f : R^n \to R$, U is an open set and unknown function is $u : \overline{U} \to R.$

Solution: We know that $x \to \Phi(x-y) f(y)$ for $x \neq y$ is harmonic for each point $y \in R^n$, and so is the sum of finitely many such expression constructed for different points y Consider the convolution

$$u(x) = \int_{R^n} \Phi(x-y) f(y) dy \qquad \ldots (5)$$

Form equations (4) and (5), we have

$$u(x) = \begin{cases} \dfrac{-1}{2\pi} \displaystyle\int_{R^n} \log(|x-y|) f(y) dy & (n=2) \\ \dfrac{1}{n(n-2)\alpha(n)} \displaystyle\int_{R^n} \dfrac{f(y)}{|x-y|^{n-2}} dy & (n \geq 3) \end{cases} \qquad \ldots (6)$$

For simplicity, we assume that the function f used in Poission's equation is twice continuously differentiable. Now, we show that $u(x)$ defined by equation (5) satisfies

(i) $u \in C^2(R^n)$

(ii) $\Delta u = -f \quad$ in R^n.

Consequently, the function in (6) provided us with a formula for a solution of Poission's equation.

Proof of (i):

Define u as,

$$u(x) = \int_{R^n} \Phi(x-y)\, f(y) dy$$

By change of variable $x - y = z$

$$u(x) = \int_{R^n} \Phi(x) f(x-z) dz = \int_{R^n} \Phi(x) f(x-y) dy$$

By definition . u_{x_i} .is

$$\frac{u(x+he_i) - u(x)}{h} = \int_{R^n} \Phi(y) \left[\frac{f(x+he_i) - f(x)}{h} \right] dy \qquad (*)$$

where $h \neq 0$ is a real number $e_i \in R^n$, $e_i = (0,0,...,0,1,0,...,0)$ with 1 in the ith position.

Thus, on taking $h \to 0$ in equation (*), we have

$$\frac{\partial u(x)}{\partial x_i} = \int_{R^n} \Phi(y) \left\{ \frac{\partial f(x-y)}{\partial x_i} \right\} dy \qquad (**)$$

for $i = 1, 2, 3, ..., n$

Similarly,

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = \int_{R^n} \Phi(y) \left\{ \frac{\partial^2 f(x-y)}{\partial x_i \partial x_j} \right\} dy \qquad (***)$$

for $i, j = 1, 2, ..., n$

As the expression on the right hand side of equation (***) is continuous in the variable x, we see that

$$u \in C^2(R^n)$$

This proves (i).

Proof of (ii)

(ii) From part (i), we have

$$\Delta u(x) = \int_{R^n} \Phi(y) \Delta_x f(x-y) dy$$

Since $\Phi(y)$ is singular at $y = 0$, so we include it in small ball $B(0, \varepsilon)$, where $\varepsilon > 0$

Then,

$$\Delta u(x) = \int_{B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y) dy + \int_{R^n - B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y) dy$$

$$= I_\varepsilon + J_\varepsilon \qquad \ldots(7)$$

where

$$I_\varepsilon = \int_{B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y) dy \qquad \ldots (8)$$

$$J_\varepsilon = \int_{R^n - B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y) dy \qquad \ldots (9)$$

Now,

$$|I_\varepsilon| = \left| \int_{B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y) dy \right|$$

$$\leq c \|D^2 f\|_{L^\infty(R^n)} \int_{B(0,\varepsilon)} |\Phi(y)| dy$$

$$\leq \begin{cases} c\varepsilon^2 |\log \varepsilon| & (n = 2) \\ c\varepsilon^2 & (n \geq 3) \end{cases}$$

Now,

$$J_\varepsilon = \int_{R^n - B(0,\varepsilon)} \Phi(y)\Delta_x f(x-y)\,dy$$

$$= \int_{R^n - B(0,\varepsilon)} \Phi(y)\Delta_y f(x-y)\,dy \qquad \left(\because \frac{\partial}{\partial x} = -\frac{\partial}{\partial y}, \Delta_x = \Delta_y\right)$$

Integrating by parts

$$J_\varepsilon = -\int_{R^n - B(0,\varepsilon)} D\Phi(y) D_y f(x-y)\,dy + \int_{\partial B(0,\varepsilon)} \Phi(y)\frac{\partial f(x-y)}{\partial \nu}\,ds(y)$$

where ν denoting the inward pointing unit normal along $\partial B(0,\varepsilon)$.

$$J_\varepsilon = K_\varepsilon + L_\varepsilon$$

We take,

$$|L_\varepsilon| = \left| \int_{\partial B(0,\varepsilon)} \Phi(y)\frac{\partial f(x-y)}{\partial \nu}\,ds(y) \right|$$

$$\le \|Df\|_{L^\infty(R^n)} \int_{\partial B(0,\varepsilon)} |\Phi(y)|\,ds(y)$$

$$|L_\varepsilon| \le \begin{cases} c\varepsilon|\log\varepsilon| & (n=2) \\ c\varepsilon & (n \ge 3) \end{cases} \qquad \dots (10)$$

Now $$K_\varepsilon = -\int_{R^n - B(0,\varepsilon)} D\Phi(y) D_y f(x-y)\,dy$$

Integrating by parts

$$K_\varepsilon = \int_{R^n - B(0,\varepsilon)} \Delta\Phi(y) f(x-y)\,dy - \int_{\partial B(0,\varepsilon)} \frac{\partial \Phi(y)}{\partial \nu} f(x-y)\,ds(y)$$

$$= -\int_{\partial B(0,\varepsilon)} \frac{\partial \Phi(y)}{\partial \nu} f(x-y)\,ds(y) \qquad \text{(since } \Phi \text{ is harmonic)}$$

$$D\Phi(y)=\left(\frac{\partial\Phi}{\partial y_1},\quad\frac{\partial\Phi}{\partial y_2},\ldots,\quad\frac{\partial\Phi}{\partial y_n}\right)$$

$$Also\ \ \frac{\partial\Phi}{\partial y_i}=\frac{\partial}{\partial y_i}\left(\frac{1}{n(n-2)\alpha(n)}\frac{1}{|y|^{n-2}}\right)$$

$$=\frac{-(n-2)}{n(n-2)\alpha(n)}\frac{1}{|y|^{n-1}}\frac{\partial|y|}{\partial y_i}=\frac{-1}{n\alpha(n)|y|^{n-1}}\frac{y_i}{|y|}$$

$$=\frac{-1}{n\alpha(n)}\frac{y}{|y|^n}\qquad\qquad (y\neq 0)$$

and $\qquad \nu=\dfrac{-y}{|y|}=\dfrac{-y}{\varepsilon}\qquad$ on $\qquad \partial B(0,\varepsilon)$

So,

$$\frac{\partial\Phi(y)}{\partial\nu}=\nu.D\Phi(y)=\frac{1}{n\alpha(n)\varepsilon^{n-1}}\qquad \text{on } \partial B(0,\varepsilon)$$

Now, we have

$$K_\varepsilon=-\frac{1}{n\alpha(n)\varepsilon^{n-1}}\int\limits_{\partial B(0,\varepsilon)} f(x-y)\,ds(y)$$

$$K_\varepsilon=-\oint\limits_{\partial B(x,\varepsilon)} f(y)\,ds(y)\to -f(y)\quad \text{as } \varepsilon\to 0 \qquad \ldots (11)$$

Combining equations (5) to (11) and letting $\varepsilon\to 0$, we have

$$\Delta u(x)=-f(x)$$

This completes the proof. Thus $u(x)$ given by (5) is the solution of Poission's equation.

2.4.2 Some Important Properties (in Polar coordinates)

(i) $\displaystyle\int\limits_{R^n} f dx=\int\limits_0^\infty\int\limits_{\partial(x,r)}(fds)\,dr$

(ii) $\displaystyle\int\limits_{B(x_0,r)} fdx=\int\limits_0^r\left(\int\limits_{\partial B(x_0,r)} fds\right)dr$

(iii) $\displaystyle\frac{d}{dr}\left(\int\limits_{B(x_0,r)} fdx\right)=\int\limits_{\partial B(x_0,r)} fds$

2.5 Mean-value Theorem

Theorem: Mean-value formulas for Laplace's equation

If u is a harmonic function. Then

$$u(x) = \oint_{\partial B(x,r)} u ds = \oint_{B(x,r)} u dy \qquad \ldots (1)$$

for each ball $B(x,r) \subset U$.

OR

If u is harmonic function, prove that u equals to both the average of u over the sphere $\partial B(x,r)$ and the average of u over the entire ball $B(x,r)$ provided $B(x,r) \subset U$.

Proof: (Proof of Part I)

Set $$\Phi(r) := \oint_{\partial B(x,r)} u(y) ds(y) \qquad \textbf{... (2)}$$

Shifting the integral to unit ball, if z is an arbitrary point of unit ball then

$$\Phi(r) := \oint_{\partial B(x,r)} u(x+rz) ds(z)$$

Then

$$\Phi'(r) = \oint_{\partial B(0,1)} Du(x+rz).z ds(z)$$

And consequently, using Green's formulas, we have

$$\Phi'(r) = \oint_{\partial B(x,r)} Du(y).\frac{y-x}{r} ds(y)$$

$= \oint_{\partial B(x,r)} Du(y).\nu ds(y)$, where ν is unit outward normal to $\partial B(x,r)$.

$$\Phi'(r) = \oint_{\partial B(x,r)} \frac{\partial u}{\partial \nu} ds(y)$$

$$= \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u(y) dy$$

$$= \frac{r}{n} \oint_{B(x,r)} \Delta u(y) dy = 0 \qquad (\because \Delta u = 0 \quad on \quad B(x,r))$$

Hence Φ is constant and

$$\Phi(r)=\lim_{t\to 0}\Phi(t)=\lim_{t\to 0}\oint_{\partial B(x,t)} u(y)\,ds(y)=u(x) \qquad \ldots(3)$$

From (2) and (3), we have

$$u(x)=\oint_{\partial B(x,r)} u(y)\,ds(y) \qquad \ldots\ (4)$$

(Proof of Part II)

Using coarea formula, we have

$$\int_{B(x,r)} u\,dy=\int_0^r\left(\int_{\partial B(x,t)} u\,ds\right)dt$$

$$=\int_0^r u(x)\,n\alpha(n)\,t^{n-1}dt$$

$$=u(x)\alpha(n)r^n$$

$$\Rightarrow u(x)=\frac{1}{\alpha(n)r^n}\int_{B(x,r)} u\,dy$$

$$=\oint_{B(x,r)} u\,dy \qquad \ldots\ (5)$$

From (4) and (5), we have

$$u(x)=\oint_{\partial B(x,r)} u\,ds=\oint_{B(x,r)} u\,dy$$

Hence proved.

Converse of Mean- value Theorem

Theorem: If $u\in C^2(U)$ satisfies the mean value formula

$$u(x)=\oint_{\partial B(x,r)} u\,ds$$

for each ball $B(x,r)\subset U$, then u is harmonic.

Proof: Suppose that u is not harmonic, so $\Delta u\neq 0$. Therefore there exists a ball $B(x,r)\subset U$ such that $\Delta u>0$ within $B(x,r)$.

But then for Φ, we know that

$$0=\Phi'(r)=\frac{r}{n}\oint_{B(x,r)} \Delta u(y)\,dy>0$$

which is a contradiction. Hence u is harmonic in U .

2.6 Properties of Harmonic Functions

Here, we present an interesting deduction about the harmonic function, all based upon the mean-value formula by assuming the following properties that $U \subset R^n$ is open and bounded.

2.6.1 Strong Maximum Principle, Uniqueness

Theorem: Let $u \in C^2(U) \cap C(\bar{U})$ is harmonic within U.

(i) Then $$\max_{\bar{U}} u = \max_{\partial U} u$$

(ii) Furthermore, if U is connected and there exists a point $x_0 \in U$ such that

$$u(x_0) = \max_{\bar{U}} u,$$

then u is constant within U.

Assertion (i) is the maximum principle for Laplace's equation and (ii) is the strong maximum principle.

Proof: (ii) Suppose there exist a point $x_0 \in U$ such that

$$u(x_0) = \max_{\bar{U}} u = M \qquad \ldots (1)$$

Then for $0 < r < dist(x_0, \partial U)$, the mean value property implies

$$\begin{aligned} M &= u(x_0) = \oint_{B(x_0,r)} udy \\ &= \frac{1}{\alpha(n)r^n} \int_{B(x_0,r)} udy \\ &\le \frac{M}{\alpha(n)r^n} \int_{B(x_0,r)} dy \\ &\le M \end{aligned}$$

Equality holds only if $u = M$ within $B(x_0, r)$. So we have, $u(y) = M$ for all $y \in B(x_0, r)$. To show that this result holds for the set U.

Consider the set

$$X = \{x \in U \,|\, u(x) = M\}$$

We prove that X is both open and closed.

X is closed since if x is the limit point of set X, then $\exists$ a sequence $\{x_n\}$ in X such that $\{x_n\} \to x$

Since u is continuous so $\{u(x_n)\} \to u(x)$.

So $$u(x) = M$$

$$\Rightarrow x \in X$$

$\Rightarrow X$ is closed.

To show that X is open, let $x \in X$, there exists a ball $B(x,r) \subset U$ such that

$$u(x) = \oint_{B(x,r)} udy$$

So $x \in B(x,r) \subset X$.

Hence X is open.

But U is connected. The only set which is both open and closed in U is itself U. So $U = X$.

Hence $u(x) = M \quad \forall x \in U$. So u is constant in U.

(i) Using above result, we have $u(y) \le u(x_0)$ for some y and suppose $x_0 \notin \partial U$.

Since U is harmonic, so by mean value theorem, there exists a ball $B(x_0, r) \subset U$ such that

$$u(x_0) = \oint_{\partial B(x_0,r)} uds(y)$$

$$M \le \frac{1}{n\alpha(n)r^{n-1}} |u(y)| \left| \int_{\partial B(x_0,r)} ds(y) \right|$$

$$\le |u(y)|$$

Maximum value is less than $|u(y)|$, which is a contradiction.

Hence $x_0 \in \partial U$.

Remarks: 1. If U is connected and $u \in C^2(U) \cap C(\bar{U})$ satisfies

$$\Delta u = 0 \text{ in } U$$

$$u = g \text{ on } \partial U$$

where $g > 0$.

Then u is positive everywhere in U if g is positive somewhere on ∂U.

2. An important application of maximum modulus principle is establishing the uniqueness of solutions to certain boundary value problem for poission's equation.

Theorem: (Uniqueness)

Let $g \in C(\partial U)$, $f \in C(U)$. Then there exists at most one solution $u \in C^2(U) \cap C(\bar{U})$ of the boundary value problem

$$-\Delta u = f \text{ in } U$$

$$u = g \text{ on } \partial U$$

Proof: Let u and $\bar{u}$ be two solutions of given boundary value problem, then

$$-\Delta u = f \text{ in } U$$

$$u = g \text{ on } \partial U$$

and

$$-\Delta \bar{u} = f \text{ in } U$$

$$\bar{u} = g \text{ on } \partial U$$

Let $w = \pm(u - \bar{u})$

$$\Delta w = 0 \text{ in } U$$

$$w = 0 \text{ on } \partial U$$

$\Rightarrow w$ is harmonic in U and w attains maximum value on boundary which is zero. If U is connected then w is constant. So $w = 0$ in U

Hence $u = \bar{u}$ in U .

2.6.2 Regularity

In this property, we prove that if $u \in C^2$ is harmonic, then necessarily $u \in C^\infty$. Thus harmonic functions are automatically infinitely differentiable.

Theorem: If $u \in C(U)$ satisfies the mean value property for each ball $B(x,r) \subset U$, then

$$u \in C^\infty(U)$$

Proof: Define a set $U_\varepsilon = \{x \in U \,|\, dist(x, \partial U) > \varepsilon\}$ and η be a standard mollifier.

Set $\quad u^\varepsilon = \eta_\varepsilon * u \quad \text{in} \quad U_\varepsilon$ … (1)

We first show that $u^\varepsilon \in C^\infty(U_\varepsilon)$.

Fix $x \in U_{\varepsilon}$, where $x = (x_1, x_2, ..., x_n)$.

Let h be very small such that $x + he_i \in U_{\varepsilon}$.

$$u^{\varepsilon}(x) = \eta_{\varepsilon} * u$$

$$= \frac{1}{\varepsilon^n} \int_{U_{\varepsilon}} \eta\left(\frac{x-y}{\varepsilon}\right) u(y)\,dy \qquad \dots (2)$$

$$u^{\varepsilon}(x + he_i) = \frac{1}{\varepsilon^n} \int_{U_{\varepsilon}} \eta\left(\frac{x - y + he_i}{\varepsilon}\right) u(y)\,dy \qquad \dots (3)$$

Now using (2) and (3), we have

$$\frac{u^{\varepsilon}(x + he_i) - u^{\varepsilon}(x)}{h} = \frac{1}{\varepsilon^n} \int_{U_{\varepsilon}} \left[\frac{\eta\left(\frac{x-y+he_i}{\varepsilon}\right) - \eta\left(\frac{x-y}{\varepsilon}\right)}{h}\right] u(y)\,dy$$

Taking the limit as $h \to 0$

$$\frac{\partial u^{\varepsilon}}{\partial x_i} = \frac{1}{\varepsilon^{n+1}} \int_{U_{\varepsilon}} \frac{\partial \eta\left(\frac{x-y}{\varepsilon}\right)}{\partial x_i} u(y)\,dy = \int_{U_{\varepsilon}} \frac{\partial \eta_{\varepsilon}(x-y)}{\partial x_i} u(y)\,dy$$

Since $\eta \in C^{\infty}(R^n)$. So $\frac{\partial u^{\varepsilon}}{\partial x_i}$ exists.

Similarly $D^{\alpha} u^{\varepsilon}$ exists for each multi-index α.

So $u^{\varepsilon} \in C^{\infty}(U_{\varepsilon})$.

We now show that $u = u^{\varepsilon}$ on U_{ε}.

Let $x \in U_{\varepsilon}$ then

$$u^{\varepsilon}(x) = \int_{U} \eta_{\varepsilon}(x-y) u(y)\,dy$$

$$= \int_{B(x,\varepsilon)} \frac{1}{\varepsilon^n} \eta\left(\frac{|x-y|}{\varepsilon}\right) u(y)\,dy$$

$$= \frac{1}{\varepsilon^n}\int_0^{\varepsilon}\eta\left(\frac{r}{\varepsilon}\right)\left(\int_{\partial B(x,r)} u(y)\,ds\right)dr \qquad \text{(using the cor. of coarea formula)}$$

$$= \frac{1}{\varepsilon^n}\int_0^{\varepsilon}\eta\left(\frac{r}{\varepsilon}\right)n\alpha(n)r^{n-1}u(x)\,dr \qquad \text{(by Mean value formula)}$$

$$= \frac{n\alpha(n)u(x)}{\varepsilon^n}\int_0^{\varepsilon}\eta\left(\frac{r}{\varepsilon}\right)r^{n-1}dr$$

$$= \frac{u(x)}{\varepsilon^n}\int_{B(0,\varepsilon)}\eta\left(\frac{y}{\varepsilon}\right)dy$$

$$= u(x)\int_{B(0,\varepsilon)}\eta_{\varepsilon}(y)\,dy \qquad \text{(by definition)}$$

$$= u(x)$$

So $u^{\varepsilon} \equiv u$ in U_{ε} and so $u \in C^{\infty}\left(U_{\varepsilon}\right)$ for each $\varepsilon > 0$.

Note: The above property holds for each $\varepsilon > 0$. It may happen u may not be smooth or even continuous upto ∂U .

2.6.3 Local Estimate for Harmonic Functions

Theorem: Suppose u is harmonic in U . Then

(i) $$\left|D^{\alpha}u\left(x_0\right)\right| \le \frac{C_k}{r^{n+k}}\|u\|_{L^1\left(B\left(x_0,r\right)\right)} \qquad \ldots (1)$$

For each ball $B(x_0,r) \subset U$ and each multiindex α of order $|\alpha| = k$.

(ii) $$C_0 = \frac{1}{\alpha(n)} \quad , \quad C_k = \frac{\left(2^{n+1}nk\right)^k}{\alpha(n)} \qquad (k = 1,\ldots) \qquad \ldots (2)$$

Proof: We prove this by induction.

For $k = 0, \alpha = 0$.

To show $\left|u\left(x_0\right)\right| \le \frac{1}{r^n\alpha(n)}\|u\|_{L^1(B(x,r))}$

By mean value theorem

$$u\left(x_0\right) = \oint_{B\left(x_0,r\right)} u(y)\,dy \quad \text{for each ball } B\left(x_0,r\right) \subset U$$

$$u(x_0) = \frac{1}{\alpha(n) r^n} \int_{B(x_0,r)} u(y)\,dy$$

$$|u(x_0)| \le \frac{1}{\alpha(n) r^n} \|u\|_{L^1(B(x_0,r))} \qquad \dots (3)$$

$$|D^0 u(x_0)| \le \frac{C_0}{r^n} \|u\|_{L^1(B(x_0,r))}$$

Hence the result.

For k=1, To show

$$|Du(x_0)| \le \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x_0,r))}$$

where $$C_1 = \frac{2^{n+1} n}{\alpha(n)}$$

Consider

$$\Delta u_{x_i} = \frac{\partial^2}{\partial x_1^2}(u_{x_i}) + \dots + \frac{\partial^2}{\partial x_n^2}(u_{x_i})$$

$$= \frac{\partial}{\partial x_i}(\Delta u) = 0$$

So, u_{x_i} is harmonic. By mean value theorem

$$|u_{x_i}(x_0)| = \left| \oint_{B\left(x_0, \frac{r}{2}\right)} u_{x_i}\,dx \right|$$

$$= \left| \frac{1}{\alpha(n)\left(\frac{r}{2}\right)^n} \int_{B\left(x_0, \frac{r}{2}\right)} u_{x_i}\,dx \right|$$

$$= \left| \frac{1}{\alpha(n)\left(\frac{r}{2}\right)^n} \int_{B\left(x_0, \frac{r}{2}\right)} u v_i\,ds \right| \qquad \text{(By Gauss- Green Theorem)}$$

$$= \left|\frac{2^n}{r}\right| \left| \oint_{\partial B\left(x_0, \frac{r}{2}\right)} u v_i ds \right|$$

$$\leq \frac{2^n}{r} \|u\|_{L^\infty\left(\partial B\left(x_0, \frac{r}{2}\right)\right)} \qquad \ldots \; (4)$$

If $x \in \partial B\left(x_0, \frac{r}{2}\right)$ then $B\left(x, \frac{r}{2}\right) \subset B(x_0, r) \subset U$.

By equation (3)

$$|u(x)| \leq \frac{2^n}{\alpha(n) r^n} \|u\|_{L^1\left(B\left(x, \frac{r}{2}\right)\right)}$$

$$\leq \frac{2^n}{\alpha(n) r^n} \|u\|_{L^1\left(B\left(x_0, r\right)\right)}$$

Hence

$$\|u\|_{L^\infty\left(\partial B\left(x_0, \frac{r}{2}\right)\right)} \leq \frac{1}{\alpha(n)}\left(\frac{2}{r}\right)^n \|u\|_{L^1\left(B\left(x_0, r\right)\right)} \qquad \ldots (5)$$

From (4) and (5)

$$\left|u_{x_i}\left(x_0\right)\right| \leq \frac{2^{n+1}.n}{\alpha(n) r^{n+1}} \|u\|_{L^1\left(B\left(x_0, r\right)\right)}$$

$$\Rightarrow \left|D^\alpha u\left(x_0\right)\right| \leq \frac{C_1}{r^{n+1}} \|u\|_{L^1\left(B\left(x_0, r\right)\right)}$$

Hence result is true for k=1.

Assume that result is true for each multiindex of order less than or equal to k-1 for all balls in U. Fix $B(x_0, r) \subset U$ and α be multiindex with $|\alpha| = k$

$$D^\alpha u = \left(D^\beta u\right)_{x_i} \text{ for some } i = (1, 2, 3, \ldots, n)$$

where $|\beta| = k-1$. Consider the ball $B\left(x_0, \frac{r}{k}\right)$

$$\left|D^{\alpha}u(x_0)\right| = \left|\left(D^{\beta}u\right)_{x_i}\right|$$

$$\leq \frac{kn}{r}\left\|D^{\beta}u\right\|_{L^{\infty}\left(\partial B\left(x_0,\frac{r}{k}\right)\right)} \qquad \ldots (6)$$

If $x \in \partial B\left(x_0, \frac{r}{k}\right)$ then

$$B\left(x, \frac{k-1}{k}r\right) \subset B(x_0, r) \subset U$$

By assumption, in the ball $B\left(x, \frac{k-1}{k}r\right)$

$$\left|D^{\beta}u(x_0)\right| \leq \frac{\left[2^{n+1}n(k-1)\right]^{k-1}}{\alpha(n)\left(\frac{k-1}{k}r\right)^{n+k-1}}\|u\|_{L^1\left(B\left(x,\frac{k-1}{k}r\right)\right)} \qquad \ldots (7)$$

From (6) and (7)

$$\left|D^{\alpha}u(x_0)\right| \leq \frac{kn}{r}\frac{\left[2^{n+1}n(k-1)\right]^{k-1}}{\alpha(n)\left(\frac{k-1}{k}r\right)^{n+k-1}}\|u\|_{L^1(B(x_0,r))}$$

$$\leq \frac{\left(2^{n+1}nk\right)^{k}}{\alpha(n)r^{n+k}}\|u\|_{L^1(B(x_0,r))}$$

Since,

$$\frac{1}{2}\left[\frac{k}{2(k-1)}\right]^{n} < 1 \quad \text{for all} \quad k \geq 2$$

Hence result holds for $|\alpha| = k$.

2.6.4 Liouville's Theorem

We see that there are no nontrivial bounded harmonic functions on all of R^n

Theorem: Suppose $u: R^n \to R$ is harmonic and bounded. Then u is constant.

Proof: Let $x_0 \in R^n, r > 0$, then by mean value theorem

$$\left|Du(x_0)\right| = \left|u_{x_i}(x_0)\right| = \left| \oint_{B\left(x_0,\frac{r}{2}\right)} u_{x_i}\, dx \right|$$

$$= \left| \frac{2^n}{\alpha(n) r^n} \int_{\partial B\left(x_0,\frac{r}{2}\right)} uvds \right| \qquad \text{(By Guass Green's theorem)}$$

$$\leq \frac{2^n}{r} \|u\|_{L^\infty\left(\partial B\left(x_0,\frac{r}{2}\right)\right)}$$

If $x \in \partial B\left(x_0, \frac{r}{2}\right)$ then $B\left(x, \frac{r}{2}\right) \subset B(x_0, r)$

$$|u(x)| \leq \frac{1}{\alpha(n)} \left(\frac{2}{r}\right)^n \|u\|_{L^1(B(x_0,r))}$$

Hence

$$\left|u_{x_i}(x_0)\right| \leq \frac{2n}{r} \left(\frac{2}{r}\right)^n \frac{1}{\alpha(n)} \|u\|_{L^1(B(x_0,r))}$$

$$= \frac{2^{n+1} n}{r^{n+1} \alpha(n)} \|u\|_{L^1(B(x_0,r))}$$

$$\leq \frac{n 2^{n+1}}{r} \|u\|_{L^\infty(R^n)} \to 0 \text{ as } r \to 0$$

Hence $Du = 0$.

So u is constant.

Theorem: Representation Formula

Let $f \in C_c^2(R^n), n \geq 3$. Then any bounded solution of $-\Delta u = f$ in R^n (1)

of the form

$$u(x) = \int_{R^n} \Phi(x-y) f(y) dy + c \qquad (x \in R^n)$$

For some constant c and $\Phi(x)$ is the solution of Laplace's equation.

Proof: Since $\Phi(x) \to 0$ as $|x| \to \infty$ for $n \geq 3$

$\Rightarrow \Phi(x)$ is bounded.

Let u be a solution of equation (1) which is represented as

$$u = \int_{R^n} \Phi(x-y)\, f(y)\, dy$$

and it is bounded.

Since $f \in C^2(R^n)$ and $\Phi(x)$ is bounded for $n \geq 3$. Let $\bar{u}$ be another bounded solution of equation (1)

Define $w = u - \bar{u}$

$$\Delta w = 0$$

and w is bounded ($\because$ difference of two bounded functions)

By Liouville's theorem

$$w = \text{constant}$$

$$\text{or } u - \bar{u} = -c$$

$$\Rightarrow \bar{u} = u + c$$

This is the required result.

Note: For n=2, $\Phi(x) = \dfrac{-1}{2\pi}\log|x|$ is unbounded as $|x| \to \infty$ and so may be

$$\int_{R^2} \Phi(x-y)\, f(y)\, dy$$

2.6.5 Analytically

Theorem: If u is harmonic in U then u is analytic in U.

Proof: Suppose that x_0 be any point in U. Firstly, we show that u can be represented by a convergent power series in some neighbourhood of x_0.

Let $\mu = \dfrac{1}{4} dist(x_0, \partial U)$

Then $M = \dfrac{1}{\alpha(n) r^n} \|u\|_{L^1(B(x_0,2r))} < \infty$ … (1)

for each $x \in B(x_0, r), B(x,r) \subset B(x_0, 2r) \subset U$

By estimates of derivatives

$$|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}$$

where $\quad c_k = \dfrac{\left(2^{n+1}nk\right)^k}{\alpha(n)}$ for each $|\alpha| = k$

So $$\left\|D^\alpha u(x_0)\right\|_{L^\infty(B(x_0,r))} \le \frac{\left(2^{n+1}nk\right)^k}{\alpha(n)r^{n+k}}\|u\|_{L^1(B(x_0,r))}$$

$$\le M\left(\frac{2^{n+1}n}{r}\right)^{|\alpha|}|\alpha|^{|\alpha|} \qquad \dots (2)$$

By Sterling formula

$$\lim_{k\to 0}\frac{k^{k+\frac{1}{2}}}{k!e^k} = \frac{1}{\sqrt{2\pi}}$$

$\Rightarrow k^k \le ck!e^k$, where c is constant.

Hence,

$$|\alpha|^{|\alpha|} \le ce^{|\alpha|}|\alpha|! \qquad \dots (3)$$

for some constant c and all multi indices α .

Furthermore, the Multinomial theorem implies

$$n^k = (1+\dots+1)^k = \sum_{|\alpha|=k}\frac{|\alpha|!}{\alpha!} \qquad \dots(4)$$

where $\quad |\alpha|! \le n^{|\alpha|}\alpha!$

Using (4) and (3) in (2)

$$\left\|D^\alpha u(x_0)\right\|_{L^\infty(B(x_0,r))} \le M\left(\frac{2^{n+1}n}{r}\right)^{|\alpha|} ce^{|\alpha|}n^{|\alpha|}\alpha!$$

$$\le Mc\left(\frac{2^{n+1}n^2e}{r}\right)^{|\alpha|}\alpha!$$

Taylor series for u at x_0 is

$$\sum_\alpha \frac{D^\alpha u(x_0)}{\alpha!}(x-x_0)^\alpha$$

The sum taken over all multiindices.

We claim that this power series converges, provided

$$|x-x_0| < \frac{r}{2^{n+2}n^3e}$$

To verify this, let us compute for each N

The remainder term is

$$R_N(x) = \sum_{|\alpha|=N}^{\infty} \frac{D^\alpha u\left(x_0 + t(x-x_0)\right)(x-x_0)^\alpha}{\alpha!}$$

For some $0 \le t \le 1$, t depending on x.

$$|R_N(x)| \le cM \sum \left(\frac{2^{n+1}n^2e}{r}\right)^N \left(\frac{r}{2^{n+2}n^3e}\right)^N$$

$$\le cM \sum_{|\alpha|=N} \left(\frac{1}{2n}\right)^N$$

$$\le \frac{cM}{2^N} \to 0 \text{ as } N \to 0$$

$\Rightarrow$ Series is converges.

So $u(x)$ is analytic in neighbourhood of x_0.

But x_0 is arbitrary point of U.

So u is analytic in U.

2.6.6 Harnack's Inequality

This inequality shows that the values of non-negative harmonic functions within open connected subset of U, are comparable.

Theorem: For each connected open set $V \subset U$, $\exists$ a positive constant c, depending only on V, such that

$$\sup_V u \le c \inf_V u \qquad \dots (1)$$

For all nonnegative harmonic functions u in U.

Thus in particular

$$\frac{1}{c}u(y) \le u(x) \le cu(y) \qquad \forall x, y \in V$$

Proof: Let $r = \frac{1}{4} dist(V, \partial U)$

Choose $x, y \in V, |x-y| \le r$. Then

$$u(x) = \oint_{B(x,2r)} udz \ge \frac{1}{\alpha(n) 2^n r^n} \int_{B(y,r)} udz$$

$$= \frac{1}{2^n} \oint_{B(y,r)} udz = \frac{1}{2^n} u(y)$$

$$\Rightarrow 2^n u(x) \ge u(y) \qquad \dots (2)$$

Interchanging the role of x and y

$$2^n u(y) \ge u(x) \qquad \dots (3)$$

Combining (2) and (3)

$$2^n u(y) \ge u(x) \ge \frac{1}{2^n} u(y) \qquad x, y \in V$$

Since V is connected, $\bar{V}$ is compact, so $\bar{V}$ can be covered by a chain of finite number of balls $\{B_i\}_{i=1}$ such that $B_i \cap B_j \ne 0$ for $i \ne j$ each of radius $\frac{r}{2}$.

Therefore,

$$u(x) \ge \frac{1}{2^{nN}} u(y) \qquad \forall x, y \in V$$

$$u(x) \ge \frac{1}{c} u(y)$$

Similarly,

$$cu(y) \ge u(x)$$

So, $$\frac{1}{c} u(y) \le u(x) \le cu(y) \qquad \forall x, y \in V$$

2.7 Green's Function:

Suppose that $U \subset R^n$ is open, bounded and ∂U is C^1. We introduced general representation formula for the solution of Poisson's equation

$$-\Delta u = f \quad \text{in } U \qquad \dots (1)$$

subjected to the prescribed boundary condition

$$u = g \quad \text{on } \partial U \qquad \dots (2)$$

Theorem: (Derivative of Green's function)

Derive the Green's function of equation (1) under the initial condition (2).

Proof: Let $u \in C^2\left(\bar{U}\right)$ is an arbitrary function and fix $x \in U$, choose $\varepsilon > 0$ so small that $B(x,\varepsilon) \subset U$ and apply Green's formula on the region $V_\varepsilon = U - B(x,\varepsilon)$ to $u(y)$ and $\Phi(y-x)$.

Then, we have

$$\int_{V_\varepsilon} \left[u(y)\Delta\Phi(y-x) - \Phi(y-x)\Delta u(y)\right] dy$$

$$= \int_{\partial V_\varepsilon} \left[u(y)\frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x)\frac{\partial u(y)}{\partial \nu}\right] ds(y)$$

where ν denoting the outer unit normal vector on ∂V_ε. Also $\Delta\Phi(x-y) = 0$ for $x \neq y$.

Then

$$-\int_{V_\varepsilon} \Phi(y-x)\Delta u(y)\, dy$$

$$= \int_{\partial U + \partial B(x,\varepsilon)} \left[u(y)\frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x)\frac{\partial u(y)}{\partial \nu}\right] ds(y) \qquad \ldots (3)$$

Now

$$\left|\int_{\partial B(x,\varepsilon)} \Phi(y-x)\frac{\partial u(y)}{\partial \nu} ds(y)\right| \leq \|Du\|_{L^\infty(\partial B(x,\varepsilon))} \left|\Phi(y-x)\right| \left|\int ds(y)\right|$$

$$\leq c\frac{1}{\varepsilon^{n-2}} n\alpha(n)\varepsilon^{n-1} = o(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0 \qquad \ldots (4)$$

Also

$$\int_{\partial B(x,\varepsilon)} u(y)\frac{\partial \Phi(y-x)}{\partial \nu} ds(y) = \int_{\partial B(0,\varepsilon)} u(y+x)\frac{\partial \Phi(y)}{\partial \nu} ds(y)$$

Now

$$D\Phi(y) = -\frac{1}{n\alpha(n)}\frac{y}{|y|^n} \qquad , y \neq 0$$

$$\nu = -\frac{y}{|y|} = \frac{-y}{\varepsilon} \quad \int_{\partial B(0,\varepsilon)} u(y+x)\frac{\partial \Phi(y)}{\partial \nu} ds(y) = \int_{\partial B(0,\varepsilon)} u(y+x)\frac{1}{n\alpha(n)\varepsilon^{n-1}} ds(y)$$

$$= \frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(x,\varepsilon)} u(y)\,ds(y)$$

$$= \quad ⨍_{\partial B(x,\varepsilon)} u(y)\,ds(y) \to u(x) \quad \text{as } \varepsilon \to 0 \qquad \ldots (5)$$

Using (4) and (5) in equation (3) and making $\varepsilon \to 0$

$$-\int_U \Phi(y-x)\Delta y dy = \int_{\partial U}\left[u(y)\frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x)\frac{\partial u}{\partial \nu}\right]ds(y) + u(x)$$

$(y \neq x)$

Thus

$$u(x) = \int_{\partial U}\left[\Phi(y-x)\frac{\partial u}{\partial \nu} - u(y)\frac{\partial \Phi(y-x)}{\partial \nu}\right]ds(y) - \int_U \Phi(y-x)\Delta u(y)\,dy \qquad \ldots (6)$$

This identity is valid for any point $x \in U$ and for any function $u \in C^2(\bar{U})$ and it gives the solution of problem defined by equation (1) and (2) provided that $u(y), \frac{\partial u}{\partial \nu}$ are known on the boundary ∂U and the value of Δu in U. But $\frac{\partial u}{\partial \nu}$ is unknown to us along the boundary. Therefore, we have to eliminate $\frac{\partial u}{\partial \nu}$ to find the solution. For it, we define a correction term formula $\phi = \phi^x(y)$ (for fixed x) given by the solution of

$$\Delta \phi^x = 0 \quad \text{in U}$$

$$\phi^x = \Phi(y-x) \quad \text{on } \partial U \qquad \ldots . (7)$$

Let us apply Green's formula once more,

$$\int_U \left[u(y)\Delta\phi^x - \phi^x \Delta u(y)\right]dy = \int_{\partial U}\left[u(y)\frac{\partial \phi^x}{\partial \nu} - \phi^x \frac{\partial u}{\partial \nu}\right]ds$$

Then we have

$$-\int_U \phi^x \Delta u(y)\,dy = \int_{\partial U}\left[u(y)\frac{\partial \phi^x}{\partial \nu} - \phi^x \frac{\partial u}{\partial \nu}\right]dx \qquad \ldots (8)$$

Adding equation (6) and (8)

$$u(x) = -\int_U \left[\Phi(y-x) - \phi^x(y)\right]\Delta u(y)\,dy - \int_{\partial U} \frac{\partial\left[\Phi(y-x) - \phi^x(y)\right]}{\partial \nu} u(y)\,dy \qquad \ldots (9)$$

Now we define Green's function for the region U as

$$G(x,y)=\Phi(y-x)-\phi^{x}(y) \qquad (x,y\in U, x\neq y) \qquad \dots (10)$$

From equation (9) and (10)

$$u(x)=-\int_{U} G(x,y)\Delta u(y)dy-\int_{\partial U} u(y)\frac{\partial G(x,y)}{\partial \nu}ds(x) \qquad \dots (11)$$

where $\frac{\partial G(x,y)}{\partial \nu}=D_y G(x,y)$. $\nu(y)$ is the outer normal derivative of G with respect to the variable y. Also we observe that equation (11) is independent of $\frac{\partial u}{\partial \nu}$.

Hence the boundary value problem given by equation (1) and (2) can be solved in term of Green's function and solution is given by equation (11) is known as **Representation formula** for Green's

Function.

Note: Fix $x\in U$. Then regarding G as a function of y, we may symbolically write

$$-\Delta G=\delta_x \text{ in } U$$

$$\text{G} = 0 \text{ on } \partial U$$

where δ_x denoting the Dirac Delta function.

2.7.1 Symmetry of Green's Function

Theorem: Show that for all $x,y\in U, x\neq y$, $G(x,y)$ is symmetric i.e. $G(x,y)=G(y,x)$.

Proof: For fix $x,y\in U, x\neq y$

Write

$$v(z)=G(x,z), w(z)=G(y,z) \qquad (z\in U)$$

Then

$$\Delta v(z)=0(z\neq x), \Delta w(z)=0(z\neq y)$$

and $\qquad w=v=0$ on ∂U.

Applying Green's formula on $V=U-\left[B(x,\varepsilon)\cup B(y,\varepsilon)\right]$ for sufficiently small $\varepsilon>0$ yields.

$$\int_{\partial B(y,\varepsilon)}\left(v\frac{\partial w}{\partial \nu}-w\frac{\partial v}{\partial \nu}\right)ds(z)=\int_{\partial B(x,\varepsilon)}\left(w\frac{\partial v}{\partial \nu}-v\frac{\partial w}{\partial \nu}\right)ds(y) \qquad \dots (1)$$

ν denoting the inward pointing unit vector field on $\partial B(x,\varepsilon)\cup\partial B(y,\varepsilon)$.

Now w is smooth near x, so

$$\left|\int_{\partial B(x,\varepsilon)} \frac{\partial w}{\partial \nu} v ds\right| \le \|Dw\|_{\partial B(x,\varepsilon)} \left|\int ds\right|$$

$$\le c\varepsilon^{n-1} \to 0 \text{ as } \varepsilon \to 0 \qquad \dots \quad (2)$$

We know that $v(z) = \Phi(z-x) - \phi^x(z)$, where ϕ^x is smooth in U .Thus

$$\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \frac{\partial v}{\partial \nu} w ds = \lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(x-z) w(z) ds = w(x)$$

Now we have

$$\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \left(w\frac{\partial v}{\partial \nu} - v\frac{\partial w}{\partial \nu}\right) ds(z) \to w(x)$$

Similarly,

$$\lim_{\varepsilon \to 0} \int_{\partial B(y,\varepsilon)} \left(v\frac{\partial w}{\partial \nu} - w\frac{\partial v}{\partial \nu}\right) ds(z) \to v(y)$$

Therefore from equation (1) , we have

$$w(x) = v(y)$$

$$\Rightarrow G(x,y) = G(y,x)$$

Hence proved.

2.7.2 Green's Function for a Half Space

Definition: If $x = (x_1,...,x_{n-1},x_n) \in R_+^n$, its reflection in the plane ∂R_+^n is the point

$$\tilde{x} = (x_1,...,x_{n-1},x_n).$$

Definition: Green's function for the half space R_+^n is

$$G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x}) \qquad (x, y \in R_+^n, x \ne y)$$

Example: Solve the boundary value problem

$$\Delta u = 0 \text{ in } R_+^n$$

$$u = g \text{ on } \partial R_+^n$$

with the help of Green's function.

Solution: Let $x, y \in R_+^n, x \neq y$.

By definition, $G(x,y) = \Phi(y-x) - \phi^x(y)$

We choose the corrector term

$$\phi^x(y) = \Phi(y-\tilde{x}) \qquad \ldots (1)$$

where $\tilde{x}$ is reflection of x w. r. t. ∂R_+^n.

Clearly $\Delta\phi^x = 0$ in R_+^n

Now

$$\Phi(y-\tilde{x}) = \frac{1}{n(n-2)\alpha(n)|y-\tilde{x}|^{n-2}}, \qquad n \geq 3$$

$$\frac{\partial\Phi}{\partial y_1}(y-\tilde{x}) = -\frac{y_1 - x_1}{n\alpha(n)|y-\tilde{x}|^n}$$

$$\frac{\partial^2\Phi}{\partial y_1^2} = -\frac{1}{n\alpha(n)|y-\tilde{x}|^n} + \frac{(y_1-x_1)^2}{\alpha(n)|y-\tilde{x}|^{n+2}}$$

$$\frac{\partial^2\Phi}{\partial y_n^2} = -\frac{1}{n\alpha(n)|y-\tilde{x}|^n} + (y_n + x_n)^2$$

Adding $\Delta\Phi(y-\tilde{x}) = 0$ on ∂R_+^n $|y-x| = (y-\tilde{x})$

So $\Phi(y-\tilde{x}) = \Phi(y-x)$

Hence both conditions are satisfied.

So, Green's function

$$G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x}) \text{ is well defined.}$$

So, using the representation formula

$$u(x) = 0 - \int_{\partial R_+^n} g(y)\frac{\partial G}{\partial \nu}(x,y)\,ds(y)$$

$$\frac{\partial G}{\partial \nu}(x,y) = DG.\hat{\nu} = -\frac{\partial G}{\partial y_n}(x,y)$$

$$\frac{\partial G}{\partial y_n} = \frac{\partial \Phi}{\partial y_n}(y-x) - \frac{\partial \Phi}{\partial y_n}(y-\tilde{x})$$

$$= -\left[\frac{y_n - x_n}{n\alpha(n)|y-x|^n} - \frac{y_n + x_n}{n\alpha(n)|y-x|^n}\right]$$

$$= \frac{2x_n}{n\alpha(n)|x-y|^n} \qquad \left(on \partial R_+^n, |y-x| = |y-\tilde{x}|\right)$$

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial R_+^n} \frac{g(y)}{|x-y|^n} ds(y) \qquad \left(x \in R_+^n\right)$$

This is the required solution and is known as Poisson's formula.

The function

$$K(x,y) = \frac{2x_n}{n\alpha(n)} \frac{1}{|x-y|^n} \qquad \left(x \in R_+^n, y \in \partial R_+^n\right)$$

is Poisson's kernel for R_+^n.

2.7.3 Green's Function for a Ball

Definition: If $x \in R^n - \{0\}$, the point $\tilde{x} = \frac{x}{|x|^2}$ is called the point dual to x with respect to $\partial B(0,1)$

Definition: Green's function for the unit ball is $G(x,y) = \Phi(y-x) - \Phi(|x|(y-\tilde{x}))$ $\left(x, y \in B(0,1), x \neq y\right)$.

Example: Solve the boundary value problem

$$\Delta u = 0 \quad \text{in} \quad B(0,1)$$

$$u = g \quad \text{on} \quad \partial B(0,1)$$

with the help of Green's function.

Solution: Fix any point $x \in B^0(0,1)$ and $y \neq x$

The Green's function is given by

$$G(x,y) = \Phi(y-x) - \Phi(y)$$

We choose $\phi^x(y) = \Phi(|x|(y-\tilde{x}))$

where $\tilde{x}$ dual of x w. r. t. $\partial B(0,1)$

As we know $\Phi(y-x)$ is harmonic. So $\Phi(y-\tilde{x})$ is also harmonic for $y \neq x$. Similarly $|x|^{2-n}\,\Phi(y-\tilde{x})$ is harmonic for $y \neq x$.

Or $\Phi(|x|(y-\tilde{x}))$ is harmonic for $y \neq x$

So, $\Delta\phi^x = 0$ in $B(0,1)$

On $\partial B(0,1)$:

$$\phi(x) = \Phi(|x|(y-\tilde{x}))$$

But

$$|x|^2\,|y-\tilde{x}|^2 = |x|^2\left\{\left(y_1 - \frac{x_1}{|x|^2}\right)^2 + \ldots + \left(y_n - \frac{x_n}{|x|^n}\right)^2\right\}$$

$$= |x|^2\left\{|y|^2 + \frac{1}{|x|^2} - \frac{2xy}{|x|^2}\right\}$$

$$= \left|x^2 + 1 - 2xy\right| \qquad (\because |y| = 1)$$

$$= |x|^2 + |y|^2 - 2xy$$

$$= |x-y|^2$$

So $\phi(x) = \Phi(|x|(y-\tilde{x})) = \Phi(y-x)$.

Hence both conditions of $\phi^x(y)$ are satisfied.

So

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-\tilde{x})) \quad \text{is well defined.}$$

Hence solution of given problem is given by

$$u(x) = -\int_{\partial B(0,1)} g(y)\frac{\partial G}{\partial \nu}\,ds(y)$$

Now on $\partial B(0,1)$

$$\frac{\partial G}{\partial \nu} = \frac{\partial G}{\partial y}.\nu\,,\ \nu \text{ being the unit normal.}$$

$$= \frac{\partial G}{\partial y}\frac{y}{|y|} = \sum \frac{\partial G}{\partial y_i} y_i \qquad (\because |y| = 1)$$

$$\frac{\partial G}{\partial y_i} = \frac{x_i - y_i}{n\alpha(n)|x-y|^n} + \frac{y_i|x|^2 - x_i}{n\alpha(n)|x-y|^n}$$

$$= \frac{y_i|x|^2 - y_i}{n\alpha(n)|x-y|^n}$$

$$\frac{\partial G}{\partial \nu} = -\frac{\left(1-|x|^2\right)}{n\alpha(n)|x-y|^n}$$

Therefore we have

$$u(x) = \int_{\partial B(0,1)} g(y)\frac{1-|x|^2}{n\alpha(n)|x-y|^n} ds(y)$$

This is the required solution.

2.7.4 Energy Methods

Theorem: (Uniqueness)

There exists at most one solution $u \in C^2(\bar{U})$ of the boundary value problem

$$-\Delta u = f \quad \text{in } U$$

$$u = g \quad \text{on } \partial U$$

where U is open, bounded and ∂U is C^1.

Proof: Let $\bar{u}$ be another solution of given problem.

Let $w = u - \bar{u}$ then $\Delta w = 0$ in U

$w = 0$ on ∂U

Consider

$$\int_U w\Delta w dx = \int_U w\left(w_{x_i}\right)_{x_i} dx$$

Integrating by parts

$$= -\int_U w_{x_i} w_{x_i} dx + \int_{\partial U} w_{x_i} w\nu ds \text{ , } \nu \text{ being the unit normal}$$

$$= -\int_U |Dw|^2 dx + 0$$

$$\Rightarrow |Dw|^2 = 0 \quad \text{in} \quad U$$

$$\Rightarrow Dw = 0 \quad \text{in} \quad \text{U}$$

$$\Rightarrow w = \text{constant in } U$$

But $w = 0$ in ∂U

Hence $w = 0$ in U

$\Rightarrow u = \bar{u}$

Hence uniqueness of solution.

Dirichlet's Principle: Let us demonstrate that a solution of the boundary value problem for Poisson's equation can be characterized as the minimize of an appropriate functional.

Thus, we define the energy functional

$$I[w] = \int_U \frac{1}{2}|Dw|^2 - wfdx$$

w belonging to the admissible set $A = \{ w \in C^2(\bar{U}) \mid w = g \ on \partial U \}$

Theorem: Let $u \in C^2(\bar{U})$ be a solution of Poisson's equation. Then

$$I[u] = \min_{w \in A} I[w] \qquad \ldots (1)$$

Conversely, if $u \in A$ satisfies (1) then u is a solution of boundary value problem

$$-\Delta u = f \quad \text{in } U$$
$$u = g \quad \text{on } \partial U \qquad \ldots (2)$$

Proof: Let $w \in A$ and u be a solution of Poisson's equation. So

$$-\Delta u = f \quad \text{in} \quad U$$

$$\Rightarrow 0 = \int_U (-\Delta u - f)(u - w)\,dx$$

$$= -\int_U \Delta u (u - w)\,dx - \int_U f(u - w)\,dx$$

Integrating by parts

$$0 = \int_U Du.D(u - w)\,dx - \int_{\partial U} (u - w) Du.vds - \int_U f(u - w)\,dx$$

$$= \int_U (Du.Du - fu)\,dx - 0 - \int_U (Du.Dw - fw)\,dx$$

$$\Rightarrow \int_U \left(|Du|^2 - fu \right) dx = \int_U (Du.Dw - fw)\, dx$$

$$\Rightarrow \int_U \left(|Du|^2 - fu \right) dx \le \int \left[\frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2 - fw \right] dx \qquad \text{(By Cauchy-Schwartz's inequality)}$$

So $\int \left[\frac{1}{2}|Du|^2 - fu \right] dx \le \int \left[\frac{1}{2}|Dw|^2 - fw \right] dx$

$$I[u] \le I[w]$$

Since $u \in A$, So

$$I[u] = \min_{w \in A} I[w]$$

Conversely, suppose that $I[u] = \min_{w \in A} I[w]$

For any $v \in C_c^\infty(U)$, define $i(\tau) = I[u + \tau v]$, $\tau \in R$

So $i(\tau)$ attains minimum for $\tau = 0$

$$i'(\tau) = 0 \quad \text{for} \quad \tau = 0$$

$$i(\tau) = \int_U \left[\frac{1}{2}|Du + \tau Dv|^2 - (u + \tau v) f \right] dx$$

$$= \int_U \left[\frac{1}{2}\left(|Du|^2 + \tau^2 |Dv|^2 \right) + \tau DuDv - (u + \tau v) f \right] dx$$

$$i'(0) = \int_U [Du.Dv - vf]\, dx$$

Integration by parts

$$0 = -\int_U v \Delta u dx + \int_{\partial U} Du.\nu ds - \int vf dx$$

$$0 = \int_U [-\Delta u - f] v dx \qquad \left[\because v \in C_c^\infty(U) \right]$$

This is true for each function $v \in C_c^\infty(U)$.

So $\Delta u = -f$ in U .

So u is a solution of Poisson's equation.

CHAPTER-3

HEAT EQUATIONS

Structure

3.1 Heat Equation – Fundamental solution

3.2 Mean value formula

3.3 Properties of solutions

3.4 Energy methods for Heat Equation

3.1 Definition: The non- homogeneous **Heat** (**Diffusion) equation** is

$$u_t - \Delta u = f\left(x,t\right) \qquad \dots (1)$$

where $x \in U \subset R^n$, $f : U \times \left[0,\infty\right) \to R$, $u : \bar{U} \times \left[0,\infty\right) \to R$, the Laplacian Δ is taken w.r.t. spatial variable x, and the function f is given while we have to solve this equation for the unknown function u.

If $f(x,t) = 0$, then the equation

$$u_t - \Delta u = 0 \qquad \dots (2)$$

is known as homogeneous heat equation.

Physical interpretation: In typical applications, the Heat equation represents the evolution in time of the density u of some quantity such as Heat, chemical concentration, etc. If $V \subset U$ is any smooth subregion, the rate of change of the total quantity within V equals the negative of the net flux through ∂V .

$$\frac{d}{dt}\int_V u dx = -\int_{\partial V} \vec{F}.\hat{\nu} ds$$

$\vec{F}$ being the flux density. Thus

$$u_t = -div\vec{F} \qquad \dots (3)$$

where V is arbitrary.

Theorem: (Fundamental Solution)

Find the fundamental solution of homogeneous Heat equation

$$u_t - \Delta u = 0 \qquad in \qquad \bar{U} \times [0,\infty) \qquad ...(1)$$

where $U \subset R^n$ is open.

Proof: It can be seen from the equation (1) that first order derivate involves w.r.t. to *t* and second order derivate w.r.t. the space variables $x_1, x_2, ..., x_n$. Consequently, if u solves the equation (1), so does $u(\lambda x, \lambda^2 t)$ for $\lambda \in R$.

So, we seek a solution of equation (1) of the form

$$u(x,t) = \frac{1}{t^{\alpha}} v\left(\frac{x}{t^{\beta}}\right) \qquad \text{…. (2)}$$

for $x \in R^n$, $t > 0$. Here, α, β are constants to be determined and the function $v : R^n \to R$ must be find.

Put $y = \frac{x}{t^{\beta}}$ in equation (2), we have

$$u(x,t) = \frac{1}{t^{\alpha}} v(y) \qquad \text{... (3)}$$

Differentiating (3) w. r. t. t and x

$$u_t = \frac{-\alpha}{t^{\alpha+1}} v(y) - \frac{\beta y Dv}{t^{\alpha+1}}$$

$$\Delta u = \frac{1}{t^{\alpha+2\beta}} \Delta v$$

Using these expression in equation (1) and simplifying

$$\alpha v(y) + \beta y Dv + \frac{1}{t^{2\beta-1}} \Delta v = 0 \qquad \text{… (4)}$$

Now, we simplify the equation (4) by putting $\beta = \frac{1}{2}$, so that the transformed equation involves the variable y only and the equation is

$$\alpha v(y) + \beta y.Dv + \Delta v = 0 \qquad \text{… (5)}$$

We seek a radial solution of equation (5) as

$$v(y) = w(r) \text{ where } r = |y| \qquad \text{….(6)}$$

where $w : R \to R$.

From equation (5) and (6), we have

$$v_{y_i} = w'(|y|)\frac{\partial |y|}{\partial y} = w'(|y|)\frac{y_i}{r} \qquad (\because |y| = r)$$

and $$v_{y_i y_i} = w''(r)\left(\frac{y_i}{r}\right)\frac{\partial r}{\partial y_i} + w'(r)\left\{\frac{1}{r} - \frac{y_i^2}{r^3}\right\}$$

$$\Delta v(y) = w'' + \omega'(r)\left(\frac{n-1}{r}\right)$$

Using value of $\Delta v(y)$ in equation (4), we get

$$w'' + \left(\frac{r}{2} + \frac{n-1}{r}\right)w' + \alpha w = 0 \qquad \ldots\ (7)$$

Now, if we set $\alpha = \frac{n}{2}$ and multiply by r^{n-1} in equation (7).

Then we have

$$\left(r^{n-1}w'\right)' + \frac{\left(r^n w\right)'}{2} = 0 \qquad \ldots\ (8)$$

Integrating equation (8)

$$r^{n-1}w' + \frac{r^n w}{2} = a \text{ , where a is a constant}$$

Assuming $\lim_{r\to\infty} w, w' = 0$, we conclude $a = 0$, so

$$w' = -\frac{1}{2}rw \qquad \ldots\ (9)$$

Integrating again, we have some constant b

$$w = be^{-r^2/4} \qquad \ldots.\ (10)$$

where b is the constant of integration.

Combining (2) and (10) and our choices for α, β , we conclude that

$$\frac{b}{t^{n/2}} e^{\frac{-|x|^2}{4t}} \text{ solves the Heat equation (1)}$$

To find b, we normalize the solution

$$\int_{R^n} u(x,t)\,dx = 1$$

$$\frac{b}{t^{n/2}} \int_{R^n} e^{-|x|^2/4t}\,dx = 1$$

$$\frac{b}{t^{n/2}}\left(2\sqrt{\pi t}\right)^n = 1$$

$$b = \frac{1}{(4\pi)^{n/2}}$$

Here the function

$$\Phi(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}; (x \in R^n, t > 0) \\ 0 \qquad , \quad (x \in R^n, t \le 0) \end{cases}$$

is called the fundamental solution of the Heat equation.

Remarks: (i) Φ is singular at the point (0,0).

(ii) Sometimes, we write $\Phi(x,t) = \Phi(|x|,t)$ to emphasise that the fundamental solution is radial in the variable r.

Theorem: Solution of Initial value problem

Solve the initial value (Cauchy) problem

$$u_t - \Delta u = 0 \qquad in \qquad R^n \times (0,\infty) \qquad ...(1)$$

$$u = g \qquad on \qquad R^n \times \{t = 0\} \qquad ...(2)$$

associated with the homogeneous Heat equation, where $g \in C(R^n) \cap L^\infty(R^n)$.

Proof: Let $\Phi(x,t) = \dfrac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}; \qquad (x \in R^n, t > 0) \qquad \ldots (3)$

be the fundamental solution of the equation (1). From earlier article, we note that $(x,t) \to \Phi(x,t)$ solves the Heat equation away from the singularity (0,0) and thus so does $(x,t) \to \Phi(x-y,t)$ for each fixed $y \in R^n$. Consequently, the convolution

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{R^n} e^{\frac{-|x-y|^2}{4t}} g(y)\,dy$$

$$= \int_{R^n} \Phi(x-y)g(y)\,dy \qquad \ldots (4)$$

Here, we will show that

(i) $u \in C^\infty(R^n \times (0,\infty))$

(ii) $u_t(x,t) - \Delta u(x,t) = 0 \qquad (x \in R^n, t > 0)$

(iii) $\lim\limits_{(x,t)\to(x^0,0)} u(x,t) = g(x^0)$ for each point $x^0 \in R^n, t > 0$

Proof: (i) Since the function $\frac{1}{t^{n/2}} e^{\frac{-|x|^2}{4t}}$ is infinitely differentiable with uniform bounded derivative of all order on $R^n \times [\delta, \infty)$ for $\delta > 0$.

So $u \in C^\infty \left(R^n \times (0, \infty)\right)$.

(ii) $u_t = \int_{R^n} \Phi_t(x-y,t) g(y) dy$

$$\Delta u = \int_{R^n} \Delta\Phi(x-y,t) g(y) dy$$

$\therefore u_t - \Delta u = 0$ (since $\Phi(x-y)$ is a solution of Heat equation)

(iii) Fix $x^0 \in R^n$. Since g is continuous, given $\varepsilon > 0, \exists\ \delta > 0$ such that $\left|g(y) - g(x^0)\right| < \varepsilon$ whenever $\left|y - x^0\right| < \delta, y \in R^n$.

Then if $\left|x - x^0\right| < \frac{\delta}{2}$

$$\begin{aligned}\left|u(x,t) - g(x^0)\right| &= \left|\int_{R^n} \Phi(x-y,t)\left[g(y) - g(x^0)\right] dy\right| \\ &\le \int_{B(x^0,\delta)} \Phi(x-y,t)\left|g(y) - g(x^0)\right| dy \\ &\quad + \int_{R^n - B(x^0,\delta)} \Phi(x-y,t)\left|g(y) - g(x^0)\right| dy \\ &= I + J \qquad \dots (5)\end{aligned}$$

Now $I \le \varepsilon \int_{R^n} \Phi(x-y,t) dy = \varepsilon$

Furthermore, if $\left|x - x^0\right| \le \frac{\delta}{2}$ and $\left|y - x^0\right| \ge \delta$ then

$$\left|y - x^0\right| \le |y - x| + \frac{\delta}{2} \le |y - x| + \frac{1}{2}\left|y - x^0\right|$$

Thus $|y - x| \ge \frac{1}{2}\left|y - x^0\right|$

Consequently

$$J \le 2\|g\|_{L^\infty} \int_{R^n - B(x^0,\delta)} \Phi(x-y,t) dy$$

$$\le \frac{c}{t^{n/2}} \int_{R^n - B(x^0,\delta)} e^{\frac{-|x-y|^2}{4t}} dy$$

$$\le \frac{c}{t^{n/2}} \int_{R^n - B(x^0,\delta)} e^{\frac{-|y-x^0|^2}{16t}} dy$$

$$= \frac{c}{t^{n/2}} \int_{\delta}^{\infty} e^{\frac{-r^2}{16t}} r^{n-1} dr \to 0 \text{ as } t \to 0^+$$

Hence, if $\left|x - x^0\right| < \frac{\delta}{2}$ and t>0 is small enough, $\left|u(x,t) - g(x^0)\right| < 2\varepsilon$.

The relation implies that

$$\lim_{\substack{(x,y)\to(x^0,0)\\ x\in R^n, t\to 0^+}} u(x,t) = g(x^0)$$

Thus, we have proved that equation *u(x,t)* given by equation (4) is the solution of the initial value problem.

Theorem: Non-homogeneous Heat Equation

Solve the initial value problem

$$\begin{aligned} &u_t - \Delta u = f \qquad && in \qquad && R^n \times (0,\infty) \\ &u = 0 && on && R^n \times \{t = 0\} \end{aligned}$$

associated with the non-homogeneous Heat equation, where $f \in C_1^2\left(R^n \times [0,\infty)\right)$ and f has compact support.

Proof:

Define u as

$$u(x,t) = \int_0^t \frac{1}{\left[4\pi(t-s)\right]^{n/2}} \int_{R^n} e^{\frac{-|x-y|^2}{4(t-s)}} f(y,s) dyds \qquad \left(x \in R^n, t > 0\right) \qquad \ldots(1)$$

$$= \int_0^t \int_{R^n} \Phi(x-y, t-s) f(y,s) dyds \qquad \ldots (2)$$

where $f \in C_1^2\left(R^n \times [0,\infty)\right)$ and f has compact support.

Then

(i) $u \in C_1^2\left(R^n \times (0,\infty)\right)$

(ii) $u_t(x,t)-\Delta u(x,t)=f(x,t) \qquad (x\in R^n, t>0)$

(iii) $\lim\limits_{(x,t)\to(x^0,0)} u(x,t)=0$ for each point $x^0\in R^n \quad (x\in R^n, t>0)$

Proof: (i) Since Φ has a singularity at (0,0) we cannot differentiate under the integral sign. Substituting the variable $x-y=0, t-s=0$ and again converting to original variable.

$$u_t(x,t)=\int_0^t\int_{R^n}\Phi(y,s)\,f(x-y,t-s)\,dyds$$

Since $f\in C^2(R^n\times[0,\infty))$ and $\Phi(y,s)$ is smooth near $s=t>0$, we compute

$$u_t(x,t)=\int_0^t\int_{R^n}\Phi(y,s)\,f_t(x-y,t-s)\,dyds$$

$$+\int_{R^n}\Phi(y,t)\,f(x-y,0)\,dy \qquad \text{(By Leibnitz's rule)}$$

$$\frac{\partial^2 u}{\partial x_i\partial x_j}(x,t)=\int_0^t\int_{R^n}\Phi(y,s)\frac{\partial^2}{\partial x_i\partial x_j}f(x-y,t-s)\,dyds \quad (i,j=1,\dots,n)$$

Thus, $u_t, D_x^2u\in C^2(R^n\times(0,\infty))$.

(ii) Now

$$u_t(x,t)-\Delta u(x,t)=\int_0^t\int_{R^n}\Phi(y,s)\left[\left(\frac{\partial}{\partial t}-\Delta_x\right)f(x-y,t-s)\right]dyds+\int_{R^n}\Phi(y,t)\,f(x-y,0)\,dy$$

$$=\int_\varepsilon^t\int_{R^n}\Phi(y,s)\left[\left(\frac{-\partial}{\partial s}-\Delta_y\right)f(x-y,t-s)\right]dyds$$

$$+\int_0^\varepsilon\int_{R^n}\Phi(y,s)\left[\left(\frac{-\partial}{\partial s}-\Delta_y\right)f(x-y,t-s)\right]dyds \quad +\int_{R^n}\Phi(y,t)\,f(x-y,0)\,dy$$

$$=I_\varepsilon+J_\varepsilon+K \qquad \dots (3)$$

Now

$$|J_\varepsilon|\le\left(\|f_t\|_{L^\infty}+\|D^2f\|_{L^\infty}\right)\int_0^\varepsilon\int_{R^n}\Phi(y,s)\,dyds<\varepsilon c$$

Also, we have

$$I_\varepsilon=\int_\varepsilon^t\int_{R^n}\left[\left(\frac{\partial}{\partial s}-\Delta_y\right)\Phi(y,s)\right]f(x-y,t-s)dyds \quad +\int_{R^n}\Phi(y,\varepsilon)\,f(x-y,t-\varepsilon)\,dy$$

$$-\int_{R^n}\Phi(y,t)\,f(x-y,0)\,dy$$

$$=\int_{R^n}\Phi(y,\varepsilon)\,f(x-y,t-\varepsilon)\,dy-K \qquad \dots (4)$$

Since Φ solves the Heat equation.

Combining (2) –(4), we have

$$u_t(x,t)-\Delta u(x,t)=\lim_{\varepsilon\to 0}\int_{R^n}\Phi(y,\varepsilon)\,f(x-y,t-\varepsilon)\,dy$$

$$=f(x,y) \qquad\qquad (x\in R^n, t>0)$$

(iii)
$$u(x,t)=\int_0^t\int_{R^n}\Phi(y,s)\,f(x-y,t-s)\,dyds$$

$$\|u\|_{L^\infty(R^n)}\le\|f\|_{L^\infty(R^n)}\int_0^t\int_{R^n}\Phi(y,s)\,dyds$$

$$=\|f\|\int_0^t ds=\|f\|t$$

Taking limit as $t\to 0$

$$\lim_{t\to 0}u(x,t)=0 \text{ for each } x\in R^n.$$

3.2 Mean-Value Formula for the Heat Equation

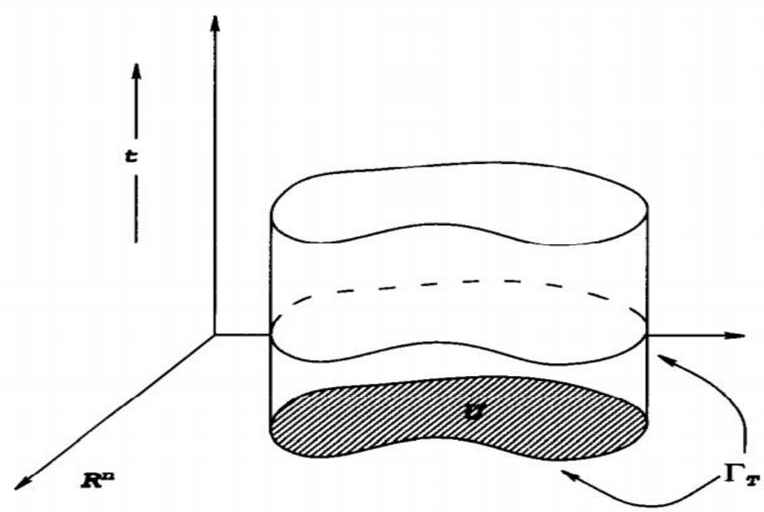
Let $U\subset R^n$ be open and bounded. Fix a time $T>0$.

Definition: The parabolic cylinder is defined as

$$U_T=U\times(0,T]$$

and the parabolic boundary of U_T is denoted by Γ_T and is defined as

$$\Gamma_T=(\bar{U}_T)-(U_T)$$



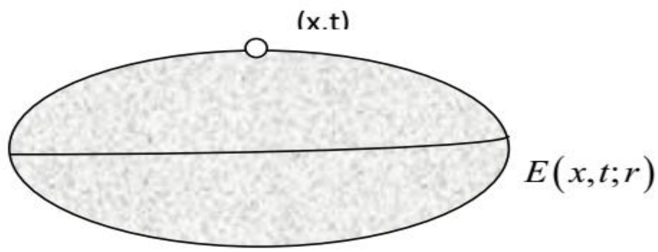
The region U_T

Interpretation: We interpret U_T as being the parabolic interior of $\bar{U}\times[0,T]$. We must note that U_T include to top $U\times\{t=T\}$. The parabolic boundary Γ_T comprises the bottom and vertical sides of $U\times[0,T]$, but not the top.

Definition (Heat ball): For fixed $x\in R^n, t\in R$ and $r>0$, we define

$$E(x,t;r)=\left\{(y,s)\in R^{n+1}\middle| s\le \text{t} \ \text{ and } \Phi(x-y,t-s)\ge\frac{1}{r^n}\right\}$$

$E(x,t;r)$ is a region in space-time. Its boundary is a level set of fundamental solutions $\Phi(x-y,t-s)$ for the Heat equation. The point (x,t) is at the centre of the top. $E(x,t;r)$ is called a Heat ball.



Heat Ball

3.2.1 Theorem: Mean-Value Property for the Heat Equation

Prove that

$$u(x,t)=\frac{1}{4r^n}\iint_{E(x,t:r)}u(y,s)\frac{|x-y|^2}{(t-s)^2}dyds \qquad \ldots (1)$$

for each Heat ball $E(x,t;r)\subset U_T$. It is assumed that $u\in C_1^2(U_T)$ solve the homogeneous Heat equation

$$u_t-\Delta u=0 \ \text{ in } \ R^n\times(0,\infty) \qquad \ldots (2)$$

Proof: The formula (1) is known as mean-value formula. We find that the right hand side of (1) involves only $u(y,s)$ for times $s\le t$. It is reasonable, as the value $u(x,t)$ should not depend upon future times. We may assume upon translating the space and time coordinates that

$$x=0, t=0 \qquad \ldots (3)$$

So we can write Heat ball as

$$E(r)=E(0,0;r) \qquad \ldots (4)$$

and set

$$\phi(r) = \frac{1}{r}\iint_{E(r)} u(y,s)\frac{|y|}{s^2}\,dyds$$

$$= \iint_{E(1)} u\left(ry, r^2 s\right)\frac{|y|^2}{s^2}\,dyds \qquad \text{(by shifting the variable)} \qquad \ldots (5)$$

Differentiating (5), we obtain

$$\phi'(r) = \iint_{E(1)} \left\{\sum_{i=1}^{n} y_i u_{y_i}\left(\frac{|y|^2}{s^2}\right) + 2ru_s\left(\frac{|y|^2}{s}\right)\right\} dyds$$

$$= \frac{1}{r^{n+1}}\iint_{E(r)} \left\{y_i u_{y_i}\left(\frac{|y|^2}{s^2}\right) + 2u_s\left(\frac{|y|^2}{s}\right)\right\} dyds \qquad \text{(Again shifting to original ball)}$$

$$= A + B \qquad \ldots (6)$$

We introduce the useful function

$$\psi = -\frac{n}{2}\log(-4\pi s) + \frac{|y|^2}{4s} + n\log r \qquad \ldots (7)$$

Then

$$\psi = 0, \qquad \text{on} \qquad \partial E(r) \qquad \ldots (8)$$

Since,

$$\Phi(y,-s) = r^{-n} \quad \text{on} \quad \partial E(r) \qquad \ldots (9)$$

be definition of Heat ball.

Now, we use (7) to write

$$B = \frac{1}{r^{n+1}}\iint_{E(r)} 4u_s \sum_{i=1}^{n} y_i \psi_{y_i}\,dyds$$

$$= -\frac{1}{r^{n+1}}\iint_{E(r)} 4nu_s\psi + 4\sum_{i=1}^{n} u_{sy_i} y_i \psi\,dyds \qquad \ldots (10)$$

There is no boundary term since $\psi = 0$ on $\partial E(r)$.

Integrating by parts with respect to s, we find

$$B = \frac{1}{r^{n+1}}\iint_{E(r)} -4nu_s\psi + 4\sum_{i=1}^{n} u_{y_i} y_i \psi_s\,dyds$$

$$= \frac{1}{r^{n+1}}\iint_{E(r)} -4nu_s\psi + 4\sum_{i=1}^{n} u_{y_i} y_i\left(\frac{-n}{2s} - \frac{|y|^2}{4s^2}\right) dyds$$

$$= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi - \frac{2n}{s}\sum_{i=1}^{n} u_{y_i} y_i dyds - A$$

This implies

$$\phi'(r) = A + B$$

$$= \frac{1}{r^{n+1}} \iint_{E(r)} \left\{ -4n\Delta u\psi - \frac{2n}{s}\sum_{i=1}^{n} u_{y_i} y_i \right\} dyds$$

$$= \sum_{i=1}^{n} \frac{1}{r^{n+1}} \iint_{E(r)} 4nu_{y_i}\psi_{y_i} - \frac{2n}{s} u_{y_i} y_i dyds \ = 0$$

Therefore, ϕ is constant.

Thus
$$\phi(r) = \lim_{t\to 0}\phi(t) = u(0,0)\left\{ \lim_{t\to 0}\frac{1}{t^n} \iint_{E(t)} \frac{|y|^2}{s^2} dyds \right\} = 4u(0,0) \qquad \ldots(11)$$

$$\frac{1}{t^n} \iint_{E(t)} \frac{|y|^2}{s^2} dyds = \iint_{E(1)} \frac{|y|^2}{s^2} dyds = 4 \qquad \ldots(12)$$

From equation (4) and (11), we write

$$u(x,t) = \frac{1}{4}\phi(r) \qquad \ldots(13)$$

From (5) and (13), we have

$$u(x,t) = \frac{1}{4r^n} \iint_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dyds \qquad \ldots (14)$$

Hence proved.

3.3 Properties of Solution

3.3.1 Theorem: Strong Maximum Principle for the Heat Equation

Assume $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ solves the Heat equation in U_T. Then

(i) $\max_{\bar{U}_T} u = \max_{\Gamma_T} u$

(ii) Furthermore, if U is connected and there exists a point $(x_0, t_0) \in U_T$ such that

$$u(x_0, t_0) = \max_{\bar{U}_T} u$$

Then u is constant in $\bar{U}_{t_0}$.

Proof: Suppose there exists a point $(x_0, t_0) \in U_T$ with

$$u(x_0,t_0) = M = \max_{\bar{U}_T} u$$

It means that the maximum value of u occur at the point (x_0,t_0).

Then for all sufficiently small r>0,

$$E(x_0,t_0;r) \subset U_T$$

By using the mean-value property, we have

$$M = u(x_0,t_0)$$

$$= \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} u(y,s) \frac{|x_0 - y|^2}{(t_0 - s)^2} dyds \leq M \qquad \ldots (1)$$

Since

$$1 = \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} \frac{|x_0 - y|^2}{(t_0 - s)^2} dyds$$

Form equation (1), it is clear that equality holds only if u is identically equal to M within $E(x_0,t_0;r)$. Consequently

$$u(y,s) = M \quad \text{for all } (y,s) \in E(x_0,t_0;r)$$

Draw any line segment L in U_T connecting (x_0,t_0) with some other point $(y_0,s_0) \in U_T$, with $s_0 < t_0$. Consider

$$r_0 = \min\{s \geq s_0 \,|\, u(x,t) = M \;\; for \;\; all \;\; point\, s(x,y) \in L, s \leq t \leq t_0\}$$

Since u is continuous, the minimum is attained. Assume $r_0 > s_0$.Then

$$u(z_0,r_0) = M$$

for some point (z_0,r_0) on $L \cap U_T$ and so

$$u \equiv M \text{ on } E(z_0,r_0;r) \text{ for all sufficiently small r>0}$$

Since $E(z_0,r_0;r)$ contains $L \cap \{r_0 - \sigma \leq t \leq r_0\}$ for some small $\sigma > 0$, which is a contradiction.

Thus

$$r_0 = s_0$$

Hence

$$u \equiv M \text{ on } L \qquad \ldots(2)$$

Now fix any point $x \in U$ and any time $0 \le t \le t_0$. There exists points $\{x_0, x_1, ..., x_m, x\}$ such that the line segments in R^n connecting x_{i-1} to x_i lie in U for $i = 1, ..., m$. (This follows since the set of points in U which can be so connected to x_0 by a polygonal path is nonempty, open and relatively closed in U). Select times $t_0 > t_1 > ... > t_m = t$. Then the line segments in R^{n+1} connecting (x_{i-1}, t_{i-1}) to $(x_i, t_i)(i = 1, ..., m)$ lie in U_T. According to step 1, $u \equiv M$ on each such segment and so $u(x,t) = M$.

Remark: 1. From a physical perspective, the maximum principle states that the temperature at any point x inside the road at any time $(0 \le t \le T)$ is less than the maximum of the initial distribution or the maximum of temperature prescribed at the ends during the time interval [0,t].

2. The strong maximum principle implies that if U is connected and $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ satisfies

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad U_T \\ u = 0 & \text{on} \quad \partial U \times [0, T] \\ u = g & \text{on} \quad U \times \{t = 0\} \end{cases}$$

where $g \ge 0$, then u is positive everywhere within U_T if g is positive somewhere on U . This is another illustration of infinite propagation speed for disturbances.

3. Similar results holds for minimum principle just by replacing “max” with “min”.

3.3.2 Theorem: Uniqueness on bounded domains

Let $g \in C(\Gamma_T)$, $f \in C(U_T)$. Then there exists at most one solution $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ of the initial/boundary-value problem

$$\begin{cases} u_t - \Delta u = f \ \ in\, U_T \\ u = g \quad on\, \Gamma_T \end{cases} \qquad \dots (1)$$

Proof: If $u = \tilde{u}$ are two solutions of (1). Then

$$\begin{cases} u_t - \Delta u = f \ \ in\, U_T \\ u = g \quad on\, \Gamma_T \end{cases} \qquad \dots (2)$$

and

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} = f \ \ in\, U_T \\ \tilde{u} = g \quad on\, \Gamma_T \end{cases} \qquad \dots (3)$$

Let $w = \pm(u - \tilde{u})$, then from equation (2) and (3), we have

$$w_t - \Delta w = (u_t - \tilde{u}_t) - \Delta(u - \tilde{u}) = 0$$

$$w = 0 \quad on \quad \Gamma_T$$

apply previous theorem to $w = \pm(u - \tilde{u})$ to get the result.

3.3.3 Regularity

Theorem: Smoothness

Suppose $u \in C_1^2(U_T)$ solves the heat equation in U_T. Then

$$u \in C^\infty(U_T)$$

This regularity assertion is valid even if u attains non-smooth boundary value on Γ_T.

Proof: We write

$$C(x,t;r) = \{(y,s) \| x - y | \le r, t - r^2 \le s \le t\}$$

To denote the closed circular cylinder of radius r, height r^2, and top centre point (x,t). Fix $(x_0,t_0) \in U_T$ and choose $r > 0$ so small that $C = C(x_0,t_0;r) \subset U_T$.

Define also the smaller cylinder

$$C' = C\left(x_0,t_0;\frac{3}{4}r\right), C'' = C\left(x_0,t_0;\frac{r}{2}\right),$$

which have the same top centre point (x_0,t_0). Extend $\zeta \equiv 0$ in $(R^n \times [0,t_0]) - C$

Assume that $u \in C^\infty(U_T)$ and set $v(x,t) = \zeta(x,t)u(x,t) \quad (x \in R^n, 0 \le t \le t_0)$

Then

$$v_t = \zeta u_t + \zeta_t u, \Delta v = \zeta \Delta u + 2D\zeta.Du + u\Delta\zeta$$

Consequently

$$v = 0 \text{ on } R^n \times \{t = 0\} \qquad \dots (1)$$

and

$$v_t - \Delta v = \zeta_t u - 2D\zeta.Du - u\Delta\zeta = \tilde{f} \quad \text{in} \quad R^n \times (0,t_0)$$

Now, set

$$\tilde{v}(x,t) = \int_0^t \int_{R^n} \Phi(x - y, t - s)\, \tilde{f}(y,s)\, dyds$$

We know that

$$\begin{cases} \tilde{v}_t - \Delta\tilde{v} = \tilde{f} \ in \ R^n \times (0,t_0) \\ \tilde{v} = 0 \quad on\, R^n \times \{t=0\} \end{cases} \qquad \ldots \ (2)$$

Since $|v|, |\tilde{v}| \le A$ for some constant A, previous theorem implies $v \equiv \tilde{v}$, i.e.

$$v(x,t) = \int_0^t \int_{R^n} \Phi(x-y,t-s)\,\tilde{f}(y,s)\,dyds$$

Now suppose $(x,t) \in C''$. As $\zeta \equiv 0$ of the cylinder C, (1) and (3) imply

$$u(x,t) = \iint_C \Phi(x-y,t-s)\left[\left(\zeta_s(y,s) - \Delta\zeta(y,s)\right)u(y,s) - 2D\zeta(y,s).Du(y,s)\right]dyds$$

Integrate the last term by parts:

$$u(x,t) = \iint_C \left[\Phi(x-y,t-s)\left(\zeta_s(y,s) + \Delta\zeta(y,s) + 2D_y\Phi(x-y,t-s).D\zeta(y,s)\right)\right]u(y,s)\,dyds$$

If u satisfies only the hypotheses of the theorem, we derive (4) with $u^\varepsilon = \eta_\varepsilon * u$ replacing u, η_ε being the standard mollifier in the variables x and t, and let $\varepsilon \to 0$.

Formula (4) has the form

$$u(x,t) = \iint_C K(x,t,y,s)\,u(y,s)\,dyds \qquad \left((x,t) \in C''\right)$$

where

$$K(x,t,y,s) = 0 \text{ for all points } (y,s) \in C'$$

Since $\zeta = 1$ on C'.

Note that K is smooth on $C - C'$.

We see u is C^∞ within $C'' = C\left(x_0, t_0; \frac{1}{2}r\right)$

Theorem: Local Estimate for Solutions of the Heat Equation

There exists for each pair of integers k, l=0,1,…, a constant $C_{k,l}$ such that

$$\max_{C\left(x,t;\frac{r}{2}\right)} \left|D_x^k D_t^l u\right| \le \frac{C_{k,l}}{r^{k+2l+n+2}} \|u\|_{L^1(C(x,t;r))}$$

for all cylinder $C\left(x,t; r/2\right) \subset C(x,t;r) \subset U_T$ and all solutions u of the Heat equation in U_T.

Proof: Fix some point in U_T. Upon shifting the coordinates, we may as well assume the point is (0,0).

Suppose first that the cylinder $C(1)=C(0,0;1)$ lies in U_T. Let $C\left(\frac{1}{2}\right)=C\left(0,0;\frac{1}{2}\right)$

Then

$$u(x,t)=\iint\limits_{C(1)} K(x,t,y,s)u(y,s)dyds \qquad \left((x,t)\in C\left(\frac{1}{2}\right)\right)$$

for some smooth function K.

Consequently,

$$\begin{aligned}\left|D_x^k D_t^l u(x,t)\right| &\le \iint\limits_{C(1)} \left|D_t^l D_x^k K(x,t,y,s)\right|\left|u(y,s)\right| dyds \\ &\le C_{kl}\,\|u\|_{L^1(C(1))}\end{aligned}$$

for some constant C_{kl}.

Now suppose the cylinder $C(r)=C(0,0;r)$ lies in U_T. Let $C\left(r/2\right)=C\left(0,0;r/2\right)$.

We define

$$v(x,t)=u\left(rx,r^2t\right)$$

Then $v_t-\Delta v=0$ in the cylinder $C(1)$.

According to (1)

$$\left|D_x^k D_t^l v(x,t)\right| \le C_{kl}\,\|v\|_{L^1(C(1))} \qquad \left((x,t)\in C\left(\frac{1}{2}\right)\right)$$

But $$D_x^k D_t^l v(x,t)=r^{2l+k}D_x^k D_t^l u\left(rx,r^2t\right)$$

and $$\|v\|_{L^1(C(1))}=\frac{1}{r^{n+2}}\|u\|_{L^1(C(r))}$$

Therefore,

$$\max_{C\left(r/2\right)}\left|D_x^k D_t^l u\right| \le \frac{C_{kl}}{r^{2l+k+n+2}}\|u\|_{L^1(C(r))}$$

Note: If u solves the Heat equation within U_T, then for each fixed time $0<t\le T$, the mapping $x\mapsto u(x,t)$ is analytic. However the mapping $t\mapsto u(x,t)$ is not in general analytic.

3.4 Energy Methods

(a) Uniqueness

Theorem: There exists at most one solution $u \in C_1^2\left(\bar{U}_T\right)$ of

$$\begin{cases} u_t - \Delta u = f \ \ in\, U_T \\ \quad u = g \quad \ on\, \Gamma_T \end{cases} \qquad \ldots \ (1)$$

Proof: If $\tilde{u}$ be another solution, $w = u - \tilde{u}$ solves

Set

$$e(t) = \int_U w^2(x,t)\,dx \qquad \left(0 \le t \le T\right)$$

Then

$$\begin{aligned} \dot{e}(t) &= 2\int_U w w_t dx \\ &= 2\int_U w \Delta w dx \\ &= -2\int_U |Dw|^2\, dx \le 0 \end{aligned}$$

and so

$$e(t) \le e(0) = 0 \qquad \left(0 \le t \le T\right)$$

Consequently $w = u - \tilde{u}$ in U_T.

(b) Backwards Uniqueness

For this, suppose u and $\tilde{u}$ are both smooth solutions of the Heat equation in U_T, with the same boundary conditions on ∂U .

$$\begin{cases} u_t - \Delta u = 0 \ in \quad U_T \\ \quad u = g \quad on\, \partial U \times [0,T] \end{cases} \qquad \ldots (1)$$

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} = 0 \ in \quad U_T \\ \quad \tilde{u} = g \quad on\, \partial U \times [0,T] \end{cases} \qquad \ldots \ (2)$$

for some function g.

Theorem: Suppose $u, \tilde{u} \in C^2\left(\bar{U}_T\right)$ solve (1) and (2). If $u(x,t) = \tilde{u}(x,t) \quad \left(x \in U\right)$ then

$$u \equiv \tilde{u} \ \text{within } U_T.$$

Proof: Write $w = u - \tilde{u}$ and set

$$e(t) = \int_U w^2(x,t)\,dx \qquad \left(0 \le t \le T\right)$$

Then

$$\dot{e}(t) = -2\int_U |Dw|^2\, dx \qquad \ldots \ (3)$$

Also

$$\ddot{e}(t) = -4\int_U Dw.Dw_t dx$$
$$= 4\int_U \Delta w w_t dx \qquad \dots (4)$$
$$= 4\int_U (\Delta w)^2 dx$$

Since w=0 on ∂U ,

$$\int_U |Dw|^2 dx = -\int_U w\Delta w dx$$
$$\leq \left(\int_U w^2 dx\right)^{1/2} \left(\int_U (\Delta w)^2 dx\right)^{1/2}$$

From (3) and (4)

$$(\dot{e}(t))^2 = 4\left(\int_U \left|Dw^2\right| dx\right)^2$$
$$\leq \left(\int_U w^2 dx\right)\left(4\int_U (\Delta w)^2 dx\right)$$
$$= e(t)\ddot{e}(t)$$

Hence

$$\ddot{e}(t)e(t) \geq (\dot{e}(t))^2 \qquad (0 \leq t \leq T) \qquad \dots (5)$$

Now if $e(t) = 0$ for all $0 \leq t \leq T$, we are done. Otherwise there exists an interval $[t_1, t_2] \subset [0, T]$ with

$$e(t) > 0 \text{ for } t_1 \leq t \leq t_2,\ e(t_2) = 0 \qquad \dots (6)$$

Write

$$f(t) = \log e(t) \qquad (t_1 \leq t \leq t_2) \qquad \dots (7)$$

Then

$$\ddot{f}(t) = \frac{\ddot{e}(t)}{e(t)} - \frac{\dot{e}(t)^2}{e(t)^2} \geq 0$$

If $0 < \tau < 1, t_1 < t < t_2$ then

$$f((1-\tau)t_1 + \tau t) \leq (1-\tau) f(t_1) + \tau f(t)$$

Also

$$e((1-\tau)t_1 + \tau t) \leq e(t_1)^{1-\tau} e(t)^{\tau},$$

and so

$$0 \leq e((1-\tau)t_1 + \tau t_2) \leq e(t_1)^{1-\tau} e(t_2)^{\tau} \quad (0 < \tau < 1)$$

This inequality implies $e(t) = 0$ for all times $t_1 \leq t \leq t_2$, a contradiction.

CHAPTER-4

WAVE EQUATIONS

Structure

4.5 Wave Equation

The homogeneous Wave equation is

$$u_{tt} - \Delta u = 0 \qquad \dots (1)$$

and the non-homogeneous Wave equation

$$u_{tt} - \Delta u = f \qquad \dots (2)$$

Here $t > 0$ and $x \in U$, where $U \subset R^n$ is open. The unknown is $u : \bar{U} \times [0, \infty) \to R, u = u(x,t)$, and the Laplacian Δ is taken with respect to the spatial variables $x = (x_1, ..., x_n)$. In equation (2) the function $f : U \times [0, \infty) \to R$ is given.

Remarks: 1. The Wave equation is a simplified model equation for a vibrating string (*n=1*). For *n=2,* it is membrane and it becomes an elastic solid for *n=3. u(x,t)* represents the displacement in some direction of the point *x* at time $t \geq 0$ for different values of *n*.

2. From physical perspective, it is obvious that we need initial condition on the displacement and velocity at time *t=0*.

Solution of Wave equation by spherical means (for n=1)

Theorem: Derive the solution of the initial value problem for one-dimensional Wave equation

$$u_{tt} - u_{xx} = 0 \text{ in } R \times (0, \infty) \qquad \dots (1)$$

$$u = g, u_t = h \text{ on } R \times \{t = 0\} \qquad \dots (2)$$

where g, h are given at time t=0..

Proof: The PDE (1) can be factored as

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = u_{tt} - u_{xx} = 0 \qquad \dots (3)$$

Set

$$v(x,t)=\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)u(x,t) \qquad \ldots (4)$$

Then, equation (4) becomes

$$v_t(x,t)+v_x(x,t)=0 \qquad (x\in R, t>0) \qquad \ldots (5)$$

Equation (5) becomes the transport equation with constant coefficient (b=1).

Let $$v(x,0)=a(x) \qquad \ldots (6)$$

We know that the fundamental solution of the initial-value problem consisting of transport equation (5) and condition (6) is

$$v(x,t)=a(x-t), x\in R, t\geq 0 \qquad \ldots (7)$$

Combining equation (4) and (7), we obtain

$$u_t(x,t)-u_x(x,t)=a(x-t) \text{ in } R\times(0,\infty) \qquad \ldots (8)$$

Also

$$u(x,0)=g(x) \text{ in } R \qquad \ldots (9)$$

By virtue of initial condition (2), Equations (8) and (9) constitute the non-homogeneous transport problem. Hence its solution is

$$u(x,t)=g(x+t)+\int_0^t a\left(x+(s-t)(-1)-s\right)ds$$

$$=g(x+t)+\frac{1}{2}\int_{x-t}^{x+t} a(y)\,dy \qquad \ldots (10) \qquad (\because x+t-2s=y)$$

The second initial condition in (2) imply

$$a(x)=v(x,0)$$

$$=u_t(x,0)-u_x(0,0)$$

$$=h(x)-g'(x), x\in R \qquad \ldots (11)$$

Substituting (11) into (10)

$$u(x,t)=g(x+t)+\frac{1}{2}\int_{x-t}^{x+t}\left[h(y)-g'(y)\right]dy$$

$$=\frac{1}{2}\left[g(x+t)+g(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy \qquad \ldots (12)$$

for $x\in R, t\geq 0$.

This is the d' Alembert's formula.

Application of d' Alembert's Formula

Initial/boundary-value problem on the half line $R_+ = \{x > 0\}$.

Example: Consider the problem

$$\begin{cases} u_{tt} - u_{xx} & in & R_+ \times (0,\infty) \\ u = g, \quad u_t = h & on & R_+ \times \{t = 0\} \\ u = 0 & on & \{x = 0\} \times (0,\infty) \end{cases} \qquad ...(1)$$

where g, h are given, with

$$g(0) = 0, h(0) = 0. \qquad \ldots (2)$$

Solution: Firstly, we convert the given problems on the half-line into the problem on whole of R We do so by extending the functions u, g, h to all of R by odd reflection method as below we set.

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & for\ x \geq 0, t \geq 0 \\ -u(-x,t) & for\ x \leq 0, t \geq 0 \end{cases} \qquad \ldots (3)$$

$$\tilde{g}(x) = \begin{cases} g(x) & for\ x \geq 0 \\ -g(x) & for\ x \leq 0 \end{cases} \qquad \ldots (4)$$

$$\tilde{h}(x) = \begin{cases} h(x) & for\ x \geq 0 \\ -h(-x) & for\ x \leq 0 \end{cases} \qquad \ldots (5)$$

Now, problem (1) becomes

$$\left.\begin{aligned} \tilde{u}_{tt} = \tilde{u}_{xx} \quad in\ R \times (0,\infty) \\ \tilde{u} = \tilde{g}, \tilde{u}_t = \tilde{h}\ on\ R \times \{t = 0\} \end{aligned}\right\} \qquad \ldots (6)$$

Hence, d' Alembert's formula for one-dimensional problem (6) implies

$$\tilde{u}(x,t) = \frac{1}{2}\left[\tilde{g}(x+t) + \tilde{g}(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)\,dy \qquad \ldots (7)$$

Recalling the definition of $\tilde{u}, \tilde{g}, \tilde{h}$ in equations (3)-(5), we can transform equation (7) to read for $x \geq 0, t \geq 0$

$$u(x,t) = \begin{cases} \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy & if\ x \geq t \geq 0 \\ \frac{1}{2}\left[g(x+t) - g(t-x)\right] + \frac{1}{2}\int_{-x+t}^{x+t} h(y)\,dy & if\ 0 \leq x \leq t \end{cases} \qquad \ldots (8)$$

Formula (8) is the solution of the given problem on the half-line $R_+ = \{x > 0\}$.

Solution of Wave Equation (for n=3)

Theorem: Derive Kirchhoff's formula for the solution of three-dimensional (n=3) initial-value problem

$$u_{tt} - \Delta u = 0 \quad \text{in} \quad R^3 \times (0,\infty) \qquad \ldots(1)$$

$$u = g \quad \text{on} \quad R^3 \times \{t=0\} \qquad \ldots(2)$$

$$u_t = h \quad \text{on} \quad R^3 \times \{t=0\} \qquad \ldots(3)$$

Solution: Let us assume that $u \in C^2\left(R^3 \times [0,\infty)\right)$ solves the above initial-value problem.

As we know

$$U(x;r,t) = \oint_{\partial B(x,r)} u(y,t)\,ds(y) \qquad \ldots(4)$$

defines the average of $u(.,t)$ over the sphere $\partial B(x,r)$. Similarly,

$$G(x;r) = \oint_{\partial B(x,r)} g(y)\,ds(y) \qquad \ldots(5)$$

$$H(x;r) = \oint_{\partial B(x,r)} h(y)\,ds(y) \qquad \ldots(6)$$

We here after regard U as a function of r and t only for fixed x.

Next, set

$$\tilde{U} = rU\,, \qquad \ldots(7)$$

$$\tilde{G} = rG, \tilde{H} = rH \qquad \ldots(8)$$

We now assert that $\tilde{U}$ solve

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 \; in & R_+ \times (0,\infty) \\ \tilde{U} = \tilde{G} \quad on & R_+ \times \{t=0\} \\ \tilde{U}_t = \tilde{H} \quad on & R_+ \times \{t=0\} \\ \tilde{U} = 0 \quad on & \{r=0\} \times (0,\infty) \end{cases} \qquad \ldots(9)$$

We note that the transformation in (7) and (8) convert the three-dimensional Wave equation into the one-dimensional Wave equation.

From equation (7)

$$\tilde{U}_{tt} = rU_{tt}$$

$$= r\left[U_{rr} + \frac{2}{r}U_r\right], \text{ Laplacian for n=3}$$

$$= rU_{rr} + 2U_r$$

$$= \left(U + rU_r\right)_r$$

$$= \left(\tilde{U}_r\right)_r = \tilde{U}_{rr} \qquad \ldots (10)$$

The problem (9) is one the half-line $R_+ = \{r \geq 0\}$.

The d' Alembert's formula for the same, for $0 \leq r \leq t$, is

$$\tilde{U}(x;r,t) = \frac{1}{2}\left[\tilde{G}(r+t) - \tilde{G}(t-r)\right] + \frac{1}{2}\int_{-r+t}^{r+t} \tilde{H}(y)\,dy \qquad \ldots (11)$$

From (4), we find

$$u(x,t) = \lim_{r \to 0^+} U(x;r,t) \qquad \ldots (12)$$

Equations (7),(8),(11) and (12) implies that

$$u(x,t) = \lim_{r \to 0^+}\left[\frac{\tilde{U}(x;r,t)}{r}\right]$$

$$= \lim_{r \to 0^+}\left[\frac{\tilde{G}(t+r) - \tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} \tilde{H}(y)\,dy\right]$$

$$= \tilde{G}'(t) + \tilde{H}(t) \qquad \ldots (13)$$

Owing then to (13), we deduce

$$u(x,t) = \frac{\partial}{\partial t}\left\{t \oint_{\partial B(x,t)} g(y)\,ds(y)\right\} + \left\{t \oint_{\partial B(x,t)} h(y)\,ds(y)\right\} \qquad \ldots (14)$$

But

$$\int_{\partial B(x,t)} g(y)\,ds(y) = \int_{\partial B(0,1)} g(x+tz)\,ds(z) \qquad \ldots (15)$$

Hence

$$\frac{\partial}{\partial t}\left\{\oint_{\partial B(x,t)} g(y)\,ds(y)\right\} = \oint_{\partial B(0,1)} \{Dg(x+tz)\}.z\,ds(z)$$

$$= \oint_{\partial B(x,t)} Dg(y).\left(\frac{y-x}{t}\right) ds(y) \qquad \ldots (16)$$

Now equation (14) and (16) conclude

$$u(x,t) = \oint_{\partial B(x,t)} \left[g(y) + \{Dg(y)\}.(y-x) + th(y) \right] ds(y) \tag{17}$$

for $x \in R^3, t > 0$.

The formula (17) is called **KIRCHHOFF'S** formula for the solution of the initial value problem (1)-(3) in 3D.

4.6 Non-Homogeneous Problem

Now we investigate the initial-value problem for the non-homogeneous Wave equation

$$\begin{cases} u_{tt} - \Delta u = f \text{ in } R^n \times (0,\infty) \\ u = 0, u_t = 0 \, on \, R^n \times \{t=0\} \end{cases} \quad \ldots (1)$$

Motivated by Duhamel's principle, which says that one can think of the inhomogeneous problem as a set of homogeneous problems each starting afresh at a different time slice $t = t_0$. By linearity, one can add up (integrate) the resulting solutions through time t_0 and obtain the solution for the inhomogeneous problem. Assume that $u = u(x,t;s)$ to be the solution of

$$\begin{cases} u_{tt}(.,s) - \Delta u(.,s) = 0 \quad in \ R^n \times (s,\infty) \\ u(.,s) = 0, u_t(.,s) = f(.,s) \, on \, R^n \times \{t=s\} \end{cases} \quad \ldots (2)$$

and set

$$u(x,t) = \int_0^t u(x,t;s)\,ds \qquad (x \in R^n, t \geq 0) \quad \ldots (3)$$

Duhamel's principle asserts that this is solution of equation (1).

Theorem: Solution of Non-homogeneous Wave Equation

Let us consider the non-homogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = f \text{ in } R^n \times (0,\infty) \\ u = 0, u_t = 0 \, on \, R^n \times \{t=0\} \end{cases} \quad \ldots (1)$$

$f \in C^{\left[n/2\right]+1}\left(R^n \times [0,\infty)\right)$ and $n \geq 2$. Define u as

$$u(x,t) = \int_0^t u(x,t;s)\,ds \qquad (x \in R^n, t \geq 0) \quad \ldots (2)$$

Then

(i) $u \in C^2\left(R^n \times [0,\infty)\right)$

(ii) $u_{tt} - \Delta u = f$ in $R^n \times (0,\infty)$

(iii) $\lim_{(x,t)\to(x^0,0)} u(x,t) = 0, \lim_{(x,t)\to(x^0,0)} u_t(x,t) = 0$ for each point $x^0 \in R^n$ $(x \in R^n, t > 0)$.

Proof: (i) If n is odd, $\left[\frac{n}{2}\right] + 1 = \frac{n+1}{2}$ and if n is even , $\left[\frac{n}{2}\right] + 1 = \frac{n+2}{2}$

Also $u(.,.;s) \in C^2(R^n \times [s,\infty))$ for each $s \ge 0$ and so $u \in C^2(R^n \times [0,\infty))$.

Hence $u \in C^2(R^n \times [0,\infty))$.

(ii) Differentiating u w.r.t t and x by two times, we have

$$u_t(x,t) = u(x,t;t) + \int_0^t u_t(x,t;s)\,ds = \int_0^t u_t(x,t;s)\,ds$$

$$u_{tt}(x,t) = u_t(x,t;t) + \int_0^t u_{tt}(x,t;s)\,ds$$

$$= f(x,t) + \int_0^t u_{tt}(x,t;s)\,ds$$

Furthermore,

$$\Delta u(x,t) = \int_0^t \Delta u(x,t;s)\,ds = \int_0^t u_{tt}(x,t;s)\,ds$$

Thus,

$$u_{tt}(x,t) - \Delta u(x,t) = f(x,t) \qquad x \in R^n, t \ge 0$$

(iii) And clearly $u(x,0) = u_t(x,0) = 0$ for $x \in R^n$. Therefore equation (2) is the solution of equation (1).

Examples: Let us work out explicitly how to solve (1) for n=1. In this case, d' Alembert's formula gives

$$u(x,t;s) = \frac{1}{2}\int_{x-t+s}^{x+t-s} f(y,s)\,dy$$

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(y,s)\,dyds$$

i.e. $$u(x,t) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)\,dyds \qquad (x \in R, t \ge 0) \qquad \ldots (5)$$

For n=3, Kirchhoff's formula implies

$$u(x,t;s)=(t-s)\oint_{\partial B(x,t-s)} f(y,s)\,dS$$

So that

$$u(x,t)=\int_0^t (t-s)\left(\oint_{\partial B(x,t-s)} f(y,s)\,dS\right)ds$$

$$=\frac{1}{4\pi}\int_0^t \int_{\partial B(x,t-s)} \frac{f(y,s)}{t-s}\,dSds$$

$$=\frac{1}{4\pi}\int_0^t \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}\,dSdr$$

Therefore,

$$u(x,t)=\frac{1}{4\pi}\int_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|}\,dy \qquad (x\in R^3, t\geq 0)$$

solves (4) for n=3.

The integrand on the right is called a retarded potential.

4.7 Energy Methods

There is the necessity of making more and more smoothness assumptions upon the data g and h to ensure the existence of a C^2 solution of the Wave equation for large and large n. This suggests that perhaps some other way of measuring the size and smoothness of functions may be more appropriate.

(a) Uniqueness

Let $U\subset R^n$ be a bounded, open set with a smooth boundary ∂U, and as usual set $U_T=U\times(0,T],\Gamma_T=\bar{U}_T-U_T$, where T>0. We are interested in the initial/boundary value problem

$$\begin{cases} u_{tt}-\Delta u=f & in \quad U_T \\ u=g & on \quad \Gamma_T \\ u_t=h & on U\times\{t=0\} \end{cases} \qquad \ldots (1)$$

Theorem: There exists at most one function $u\in C^2(\bar{U}_T)$ solving (1).

Proof: If $\tilde{u}$ is another such solution, then $w:=u-\tilde{u}$ solves

$$\begin{cases} w_{tt}-\Delta w=0 & in \quad U_T \\ w=0 & on \quad \Gamma_T \\ w_t=0 & on U\times\{t=0\} \end{cases}$$

Set "energy"

$$e(t)=\frac{1}{2}\int_U w_t^2(x,t)+|Dw(x,t)|^2\,dx \qquad (0\le t\le T)$$

Differentiating e(t), we have

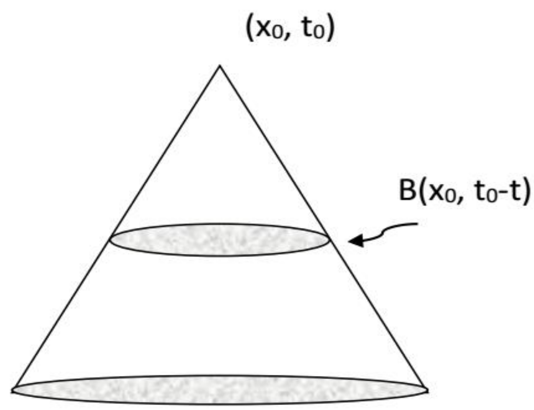
$$\dot{e}(t)=\int_U w_t w_{tt}+Dw.Dw_t dx$$

$$=\int_U w_t\left(w_{tt}-\Delta w\right)dx=0$$

There is no boundary term since w=0, and hence $w_t=0$, on $\partial U\times[0,T]$. Thus for all $0\le t\le T, e(t)=e(0)=0,$ and so $w_t, Dw=0$ within U_T. Since $w\equiv 0$ on $U\times\{t=0\}$, we conclude $w=u-\tilde{u}=0$ in U_T.

(b) Domain of Dependence

As another illustration of energy methods, let us examine again the domain of dependence of solutions to the Wave equation in all of space.



Cone of dependence

For this, suppose $u\in C^2$ solves

$$u_{tt}-\Delta u=0 \text{ in } R^n\times(0,\infty)$$

Fix $x_0\in R^n, t_0>0$ and consider the cone

$$C=\left\{(x,t)\big|0\le t\le t_0, |x-x_0|\le t_0-t\right\}.$$

CHAPTER-5

NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

Structure

5.1 Definition: Let U is an open sunset of R^n, $x = (x_1, ..., x_n) \in R^n$ and let $u : \bar{U} \subseteq R^n \to R$. A general form of first-order partial differential equation for $u = u(x)$ is given by

$$F(Du, u, x) = 0, \qquad \dots (1)$$

where $F : R^n \times R \times \bar{U} \to R$ is a given function, Du is the vector of partial derivatives of u and $u(x)$ is the unknown function.

We can write equation (1) as

$$\begin{aligned} F &= F(p, z, x) \\ &= F(p_1, p_2 ..., p_n, z, x_1, x_2, ..., x_n) \end{aligned}$$

for $p \in R^n$, $z \in R$, $x \in U$.

Here, "p" is the name of the variable for which we substitute the gradient Du and "z" is the variable for which we substitute $u(x)$. We also assume hereafter that F is smooth, and set

$$\begin{aligned} D_p F &= \left(F_{p_1}, F_{p_2}, ..., F_{p_n}\right) \\ D_z F &= F_z \\ D_x &= \left(F_{x_1}, F_{x_2}, ..., F_{x_n}\right) \end{aligned}.$$

Remark: The PDE $F(Du,u,x)=0$ is usually accompanied by a boundary condition of the form $u=g$ on ∂U. Such a problem is usually called a boundary value problem. Here our main concern is to search solution for the non-linear PDE

Complete Integral: Consider the non-linear first order PDE

$$F(Du,u,x)=0 \qquad \ldots (1)$$

Suppose first that $A\subset R^n$ is an open set. Assume for each parameter $a=(a_1,...,a_n)\in A$, we have a C^2 solution

$$u=u(x;a) \qquad \ldots (2)$$

of the PDE (1) and

$$\left(D_a u, D_{xa}^2 u\right)=\begin{bmatrix} u_{a_1} & u_{x_1a_1} & \cdots & u_{x_na_1} \\ u_{a_2} & u_{x_1a_2} & \cdots & u_{x_na_2} \\ \cdots & \cdots & \cdots & \cdots \\ u_{a_n} & u_{x_1a_n} & \cdots & u_{x_na_n} \end{bmatrix} \qquad \ldots (3)$$

A C^2 function $u=u(x;a)$ (shown in equation (2)) is called a complete integral in $U\times A$ provided

(i) $u(x;a)$ solves the PDE(1) for each $a\in A$

(ii) $rank\left(D_a u, D_{xa}^2 u\right)=n \quad (x\in U, a\in A)$

Note: Condition (ii) ensures $u(x;a)$"depends on all the n independent parameters $a_1,...,a_n$".

Example 1: The eikonal equation,

$$|Du|=1 \qquad \ldots (4)$$

Introduced by Hamilton in 1827 is an approximation to the equations which govern the behaviour of light travelling through varying materials. A solution, depending on parameters $\|a\|=1, b\in R$ is

$$u(x;a,b)=a.x+b \qquad \ldots (5)$$

Example 2: The Clairaut's equation is the PDE

$$x.Du+f(Du)=u \qquad \ldots (6)$$

where $f:R^n\to R$ is given.

A complete integral is

$$u(x;a)=a\bullet x+f(a) \quad (x\in U) \qquad \ldots (7)$$

for $a\in R^n$.

Example 3: The Hamilton-Jacobi Equation

$$u_t + H(Du) = 0 \qquad \dots (8)$$

with $H : R^n \to R$ is given and $u = u(x,t) : R^n \times R \to R$ .A solution depending on parameters $a \in R^n, b \in R$ is

$$u(x,t;a,b) = a \bullet x - tH(a) + b \qquad \dots (9)$$

where $t \geq 0$.

Remark: For simplicity, in most of what follows, we restrict to $n = 2$. We call the two variables x, y. Thus, we reduce to the case

$$F(u_x, u_y, u, x, y) = 0 \qquad \dots (7)$$

In this case, the solution $u = u(x, y)$ is a surface in R^3. The normal direction to the surface at each point is given by the vector $(u_x, u_y, -1)$.

5.2 Envelope

Definition: Let $u = u(x;a)$ be a C^1 function of x and U and A are open subsets of R^n. Consider the vector equation

$$D_a u(x;a) = 0 \quad (x \in U, a \in A) \qquad \dots (1)$$

Suppose that we can solve (1) for the parameter a as a C^1 function of x,

$$a = \phi(x) \qquad \dots (2)$$

Thus

$$D_a u(x;\phi(x)) = 0 \quad (x \in U) \qquad \dots (3)$$

We can call

$$v(x) := u(x;\phi(x)) \quad (x \in U) \qquad \dots (4)$$

is the envelope of the function $\{u(.;a)\}_{a \in A}$

Remarks: We can build new solution of nonlinear first order PDE by forming envelope and such types of solutions are called singular integral of the given PDE.

Theorem: Construction of new solutions

Suppose for each $a \in A$ as above that $u = u(.;a)$ solves the partial differential equation

$$F(Du, u, x) = 0 \qquad \dots (5)$$

Assume further that the envelope v, defined (3) and (4) above, exists and is a C^1 function. Then v solves (5) as well.

Proof: We have $v(x)=u(x;\phi(x))$

$$v_{x_i}(x)=u_{x_i}(x;\phi(x))+\sum_{j=1}^{m}u_{a_j}(x,\phi(x))\phi_{x_i}^{j}(x)$$

$$=u_{x_i}(x;\phi(x))$$

for $i=1,\dots,n$.

Hence for each $x\in U$,

$$F(Dv(x),v(x),x)=F(Du(x;\phi(x)),u(x;\phi(x)),x)=0$$

Note: The geometric idea is that for each $x\in U$, the graph of v is tangent to the graph of $u(.;a)$ for $a=\phi(x)$. Thus $Dv=D_x u(.;a)$ at x, for $a=\phi(x)$.

Example 4: Consider the PDE

$$u^2\left(1+|Du|^2\right)=1 \qquad \dots (6)$$

The complete integral is

$$u(x,a)=\pm\left(1-|x-a|^2\right)^{1/2} \qquad (|x-a|<1)$$

We find that

$$D_a u=\frac{\mp(x-a)}{\left(1-|x-a|^2\right)^{1/2}}=0$$

provided $a=\phi(x)=x$.

Thus $v\equiv\pm1$ are singular integrals of (6).

5.3 Characteristics

Theorem: Structure of Characteristics PDE

Let $u\in C^2(U)$ solves the non-linear PDE

$$F(Du,u,x)=0 \text{ in } U$$

Assume $\bar{x}(.)=(x^1,x^2,\dots,x^n)$ solves the ODE $\dot{\bar{x}}=D_pF\left(\overline{p(s)},z(s),\overline{x(s)}\right)$,

where

$\overline{p(s)} = Du\left(\overline{x}(.)\right), \qquad z(s) = u(\overline{x}(.))$

Then $\overline{p}(.)$ solves the ODE.

$\dot{\overline{p}} = -D_X F\left(\overline{p(s)}, z(s), \overline{x(s)}\right) - D_Z F\left(\overline{p(s)}, z(s), \overline{x(s)}\right)\overline{p(s)}$ (3)

and *z(s)* solves the ODE $\dot{z}(s) = D_p F\left(\overline{p(s)}, z(s), \overline{x(s)}\right).\overline{p(s)}$ for those *s* such that $\overline{x}(s) \in U$

Proof: Consider nonlinear first order PDE

$$F\left(Du, u, x\right) = 0 \text{ in } U \qquad \dots (1)$$

subject now to the boundary condition

$$u = g \quad \text{on } \Gamma \qquad \dots (2)$$

where $\Gamma \subseteq \partial U$ and $g : \Gamma \to R$ are given.

We suppose that F and g are smooth functions. Now we derive the method of characteristics which solves (1) and (2) by converting PDE into appropriates system of ODE. Initially, we would like to calculate u(x) by finding some curve lying within U, connecting x with a point $x_0 \in \Gamma$ and along which we can calculate u. Since equation (2) says $u = g$ on Γ. So we know the value of u at one end x_0 and we hope then to able to find the value of u all along the curve, and also at the particular point x.

Let us suppose the curve is described parametrically by the function

$\overline{x}(s) = \left(x^1(s), ..., x^n(s)\right)$, the parameter s lying in some subinterval of R

Assuming u is a C^2 solution of (1), we define

$$z(s) = u\left(\overline{x}(s)\right) \qquad \dots (3)$$

Set

$$\overline{p}(s) = Du\left(\overline{x}(s)\right) \qquad \dots (4)$$

i.e. $\overline{p}(s) = \left(p^1(s), ..., p^n(s)\right)$, where

$$p^i(s) = u_{x_i}\left(\overline{x}(s)\right) \quad \left(i = 1, ..., n\right). \qquad \dots (5)$$

So $z(.)$ gives the values of u along the curve and $\overline{p}(.)$ records the values of the gradient Du .

First we differentiate (5)

$$\dot{p}^i(s) = \sum_{j=1}^{n} u_{x_i x_j}\left(\overline{x}(s)\right)\dot{x}^j(s) \qquad \dots (6)$$

where $\bullet = \frac{d}{ds}$

We can also differentiate the PDE (1) with respect to x

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(Du,u,x)u_{x_j x_i}\left(\bar{x}(s)\right)\dot{x}^j(s)+\frac{\partial F}{\partial z}(Du,u,x)u_{x_i}+\frac{\partial F}{\partial x_i}(Du,u,x)=0 \quad \ldots (7)$$

We set

$$\dot{x}^j(s)=\frac{\partial F}{\partial p_j}(\bar{p}(s),z(s),\bar{x}(s)) \qquad (j=1,2,\ldots,n) \qquad \ldots(8)$$

Assuming (8) holds, we evaluate (7) at $x=\bar{x}(s)$ and using equations (3) and (4), we have the identity

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}\left(\bar{p}(s),z(s),\bar{x}(s)\right)u_{x_i x_j}\left(\bar{x}(s)\right)+\frac{\partial F}{\partial z}\left(\bar{p}(s),z(s),\bar{x}(s)\right)p^i(s)+\frac{\partial F}{\partial x_i}\left(\bar{p}(s),z(s),\bar{x}(s)\right)=0$$

Put this expression and (8) into (6)

$$\dot{p}^i(s)=-\frac{\partial F}{\partial x_i}\left(\bar{p}(s),z(s),\bar{x}(s)\right)-\frac{\partial F}{\partial z}\left(\bar{p}(s),z(s),\bar{x}(s)\right)p^i(s) \qquad \ldots(9)$$

Lastly, we differentiate (3)

$$\dot{z}(s)=\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\left(\bar{x}(s)\right)\dot{x}^j(s)=\sum_{j=1}^{n}p^j(s)\frac{\partial F}{\partial p_j}\left(\bar{p}(s),z(s),\bar{x}(s)\right) \qquad \ldots(10)$$

the second equality holding by (5) and (8). We summarize by rewriting equation (8)-(10) in vector notation

$$\begin{aligned}
\dot{\bar{p}}(s) &= -D_x F\left(\bar{p}(s),z(s),\bar{x}(s)\right)-D_x F\left(\bar{p}(s),z(s),\bar{x}(s)\right).\bar{p}(s)\\
\dot{z}(s) &= D_p F\left(\bar{p}(s),z(s),\bar{x}(s)\right).\bar{p}(s) \qquad \ldots(11)\\
\dot{\bar{x}}(s) &= D_p F\left(\bar{p}(s),z(s),\bar{x}(s)\right)
\end{aligned}$$

This system of 2n+1 first order ODE comprises the **characteristic equation of the nonlinear first order PDE (1).**

The functions $\bar{p}(.)=\left(p^1(.),\ldots,p^n(.)\right), z(.), \bar{x}(.)=\left(x^1(.),\ldots,x^n(.)\right)$ are called the **characteristics.**

Remark: The characteristics ODE are truly remarkable in that they form a closed system of equations for $\bar{x}(.), z(.)=u\left(\bar{x}(.)\right)$ and $\bar{p}(.)=Du\left(\bar{x}(.)\right)$, whenever u is a smooth solution of the general nonlinear PDE(1).

We can use $X(s)$ in place of $\bar{x}(s)$.

Now we discuss some special cases for which the structure of characteristics equations is especially simple.

(a) Article

Let us consider the PDE of the form $F(Du,u,x)=0$ to be linear and homogeneous and thus has the form

$$F(Du,u,x)=b(x).Du(x)+c(x)u(x)=0 \qquad (x\in U) \qquad \ldots(1)$$

Equation (1) can be written as

$$F(p,z,x)=b(x).p+c(x)z$$

So characteristics equations are

$$\dot{\bar{x}}(s)=D_pF=b(x)$$

$$=b(\bar{x}(s)) \qquad \text{(From last expression)}$$

and $$\dot{z}(s)=D_pF.\bar{p}=b(\bar{x}(s)).\bar{p}(s) \qquad \text{(From last expression)}$$

$$=-c(\bar{x}(s))z(s)$$

Thus

$$\begin{cases} \dot{\bar{x}}(s)=b(\bar{x}(s)) \\ \dot{z}(s)=-c(\bar{x}(s))z(s) \end{cases} \qquad \ldots(2)$$

comprise the characteristics equations for the linear first order PDE(1).

Example 5: Solve two dimensional system

$$\begin{cases} x_1u_{x_2}-x_2u_{x_1}=u \text{ in } U \\ u=g \qquad \text{on } \Gamma \end{cases} \qquad \ldots(3)$$

where U is the quadrant $\{x_1>0,x_2>0\}$ and $\Gamma=\{x_1>0,x_2=0\}\subseteq\partial U$.

Solution: Comparing (3) with (1), we have

$$F(Du,u,x)=x_1u_{x_2}-x_2u_{x_1}-u=0$$

$$\Rightarrow(-x_2,x_1).(u_{x_1},u_{x_2})-u=0$$

We get,

$$b(\bar{x}(s))=(-x_2,x_1), \qquad c(\bar{x}(s))=-1$$

Now $$b(\bar{x}(s))=(b_1(x),b_2(x))$$

$$= \left(-x_2, x_1\right)$$

$$\Rightarrow b_1\left(x\right) = -x_2, b_2\left(x\right) = x_1$$

The characteristics equations are

$$\dot{X}\left(s\right) = b\left(X\left(s\right)\right)$$

and $$\dot{z}\left(s\right) = -c\left(X\left(s\right)\right)z\left(s\right)$$

Therefore $$\dot{z}\left(s\right) = z\left(s\right)$$

$$\dot{X}\left(s\right) = \left(-x_2\left(s\right), x_1\left(s\right)\right)$$

$$\Rightarrow \left(\dot{x}_1\left(s\right), \dot{x}_2\left(s\right)\right) = \left(-x_2\left(s\right), x_1\left(s\right)\right)$$

$$\Rightarrow \dot{x}_1\left(s\right) = -x_2\left(s\right) \text{and } \dot{x}_2\left(s\right) = x_1\left(s\right) \qquad \ldots(4)$$

Now $$\ddot{x}_1\left(s\right) = -\dot{x}_2\left(s\right) = -x_1\left(s\right)$$

$$\Rightarrow \ddot{x}_1\left(s\right) + x_1\left(s\right) = 0$$

Auxiliary equation is $D^2 + 1 = 0$

$$\Rightarrow D = \pm i$$

$$\Rightarrow x_1\left(s\right) = c_1 \cos s + c_2 \sin s \qquad \ldots (5)$$

So $$\dot{x}_2\left(s\right) = c_1 \cos s + c_2 \sin s \qquad \ldots(6)$$

Integrate (5) w.r.t.s

$$x_2\left(s\right) = c_1 \sin s - c_2 \cos s + c_3 \qquad \ldots(7)$$

From (5), we have

$$\dot{x}_1\left(s\right) = -c_1 \sin s + c_2 \cos s \qquad \ldots(8)$$

Comparing (4) and (8)

$$-x_2\left(s\right) = -c_1 \sin s + c_2 \cos s$$

$$\Rightarrow x_2\left(s\right) = c_1 \sin s - c_2 \cos s \qquad \ldots(9)$$

From (7) and (9)

$$c_3 = 0$$

Therefore $$x_2(s) = c_1 \sin s - c_2 \cos s \qquad \ldots(10)$$

Taking $s = 0$ in (10)

$$x_2(0) = -c_2$$

$$\Rightarrow c_2 = 0 \qquad \left[\Gamma = \left\{\left(x_1(s), x_2(s)\right) \middle| x_2 = 0\, at \quad s = 0\right\}\right]$$

Therefore $$x_1(s) = c_1 \cos s \qquad \ldots(11)$$

and $$x_2(s) = c_1 \sin s \qquad \ldots (12)$$

Put $s = 0$ in (11)

$$x_1(0) = c_1$$

Let $x^0 = x_1(0) = c_1$

Put value of $c_1 = x^0$ in (11) and (12)

$$x_1(s) = x^0 \cos s$$
$$x_2(s) = x^0 \sin s$$

Also we have

$$\dot{z}(s) = z(s)$$
$$\Rightarrow \frac{dz}{ds} = z(s)$$

Integrating w.r.t.s

$$\log z = s + \log z^0$$
$$\Rightarrow \log \frac{z}{z^0} = s$$
$$\Rightarrow z = z^0 e^s$$
$$\Rightarrow z(0) = z^0$$

Therefore $$z(s) = z(0) e^s$$

Also $$u = g\ on \quad \Gamma$$

$$\Rightarrow u\left(x(s), 0\right) = g\left(x_1(s)\right) \qquad \ldots(13)$$

We know that $u\left(x(s)\right) = z(s)$

So
$$u\left(x_1(s),x_2(s)\right)=z(s)$$
$$\Rightarrow u\left(x_1(0),0\right)=z(0)=z^0 \qquad \ldots(14)$$

Put (14) in (13)
$$z(s)=g\left(x^0\right)e^s$$

Thus we have
$$x_1(s)=c_1\cos s=x^0\cos s$$
and
$$x_2(s)=c_1\sin s=x^0\sin s$$
and
$$z(s)=g\left(x^0\right)e^s$$

Now select s>0 and $x^0>0$, so that
$$\left(x_1,x_2\right)=\left(x_1(s),x_2(s)\right)=\left(x^0\cos s,x^0\sin s\right)$$
$$\Rightarrow x_1=x^0\cos s \; and \quad x_2=x^0\sin s$$

Consider,
$$x_1^2+x_2^2=x^{0^2}\left(\sin^2 s+\cos^2 s\right)=x^{0^2}$$
$$\Rightarrow \sqrt{x_1^2+x_2^2}=x^0$$

We have
$$\tan s=\frac{x_2}{x_1}$$
$$\Rightarrow s=\tan^{-1}\left(\frac{x_2}{x_1}\right)$$

Thus
$$u\left(x(s)\right)=z(s)=g\left(x^0\right)e^s$$
$$\Rightarrow u\left(x(s)\right)=g\left(\sqrt{x_1^2+x_2^2}\right)e^{\arctan\left(\frac{x_2}{x_1}\right)}$$

which is the required solution.

(b) Article

A quasilinear PDE is of the form
$$F\left(Du,u,x\right)=b\left(x,u(x)\right).Du(x)+c\left(x,u(x)\right)=0 \qquad \ldots\ (1)$$

Equation (1) can be written as

$$F(p,z,x)=b(x,z).p+c(x,z)$$

Now $$D_pF=b(x,z)$$

Thus the characteristic equations becomes

$$\dot{X}(s)=D_pF=b(X(s),z(s))$$

and $$\dot{z}(s)=D_pF.\overline{p}$$

$$=b(X(s),z(s)).\overline{p}(s)$$
$$=-c(X(s),z(s))$$

Consequently

$$\begin{cases}\dot{X}(s)=b(X(s),z(s))\\ \dot{z}(s)=-c(X(s),z(s))\end{cases} \qquad ..(2)$$

are the characteristic equations for the quasilinear first order PDE (1).

Example 6: Consider a boundary-value problem for a semilinear PDE

$$\begin{cases}u_{x_1}+u_{x_2}=u^2 \ in\,U\\ \quad u=g \qquad on\,\Gamma\end{cases} \qquad \ldots(3)$$

where U is half-space $\{x_2>0\}$ *and* $\Gamma=\{x_2=0\}=\partial U$.

Solution: Comparing (3) with (1), we have

$$b=(1,1) \text{ and } c=-z^2$$

Then (2) becomes

$$\begin{cases}\dot{x}^1=1,\dot{x}^2=1\\ \quad \dot{z}=z^2\end{cases}$$

Consequently

$$\begin{cases}x^1(s)=x^0+s,x^2(s)=s\\ z(s)=\dfrac{z^0}{1-sz^0}=\dfrac{g(x^0)}{1-sg(x^0)}\end{cases}$$

where $x^0 \in R, s \geq 0$, provided the denominator is not zero.

Fix a point $(x_1, x_2) \in U$. We select $s > 0$ and $x^0 \in \boldsymbol{R}$, so that $(x_1, x_2) = \left(x^1(s), x^2(s)\right) = \left(x^0 + s, s\right)$

i.e. $x^0 = x_1 - x_2, s = x_2$.

Then

$$u(x_1, x_2) = u\left(x^1(s), x^2(s)\right) = z(s) = \frac{g\left(x^0\right)}{1 - sg\left(x^0\right)}$$

$$= \frac{g(x_1 - x_2)}{1 - x_2 g(x_1 - x_2)}, 1 - x_2 g(x_1 - x_2) \neq 0$$

which is the required solution.

(c) In this case, we will discuss about characteristics equation of fully nonlinear PDE.

Example 7: Consider the fully nonlinear problem

$$\begin{cases} u_{x_1} u_{x_2} = u \ in \, U \\ u = x_2^2 \quad on \, \Gamma \end{cases} \qquad ..(1)$$

where $U = \{x_1 > 0\}, \Gamma = \{x_1 = 0\} = \partial U$

Here $F(p, z, x) = p_1 p_2 - z$. Then the characteristic equations becomes

$$\begin{cases} \dot{p}^1 = p^1, \dot{p}^2 = p^2 \\ \dot{z} = 2p^1 p^2 \\ \dot{x}^1 = p^2, \dot{x}^2 = p^1 \end{cases}$$

We integrate these equations and we find

$$\begin{cases} x^1(s) = p_2^0\left(e^s - 1\right), x^2(s) = x^0 + p_1^0\left(e^s - 1\right) \\ z(s) = z^0 + p_1^0 p_2^0\left(e^{2s} - 1\right) \\ p^1(s) = p_1^0 e^s, p^2(s) = p_2^0 e^s \end{cases}$$

Since $u = x_2^2$ on Γ, $p_2^0 = u_{x_2}\left(0, x^0\right) = 2x^0$.

Therefore, the PDE $u_{x_1} u_{x_2} = u$ itself implies $p_1^0 p_2^0 = z^0 = \left(x^0\right)^2$, and so $p_1^0 = \dfrac{x^0}{2}$.

Thus we have,

$$\begin{cases} x^1(s) = 2x^0\left(e^s - 1\right), x^2(s) = \dfrac{x^0}{2}\left(e^s + 1\right) \\ z(s) = \left(x^0\right)^2 e^{2s} \\ p^1(s) = \dfrac{x^0}{2}e^s, \; p^2(s) = 2x^0 e^s \end{cases}$$

Fix a point $(x_1, x_2) \in U$. Choose s and x^0 so that $(x_1, x_2) = (x^1(s), x^2(s)) = (2x^0(e^s - 1), \frac{x^0}{2}(e^s + 1))$

and so

$$u(x_1, x_2) = u\left(x^1(s), x^2(s)\right) = z(s) = \left(x^0\right)^2 e^{2s}$$

$$= \frac{\left(x_1 + 4x_2\right)^2}{16}$$

Exercise:

1. Find the characteristics of the following equations:
   (a) $x_1 u_{x_1} + x_2 u_{x_2} = 2u, \; u(x_1, 1) = g(x_1)$
   (b) $u_t + b.Du = f \quad in \quad R^n \times (0, \infty), b \in R^n, \; f = f(x,t)$
2. Prove that the characteristics for the Hamiltonian-Jacobi equation
   $u_t + H(Du, x) = 0$
   are

$$\dot{\overline{p}}(s) = -D_x H\left(\overline{p}(s), \overline{x}(s)\right)$$

$$\dot{z}(s) = D_p H\left(\overline{p}(s), \overline{x}(s)\right).\overline{p}(s) - H\left(\overline{p}(s), \overline{x}(s)\right)$$

$$\dot{\overline{x}}(s) = D_p H\left(\overline{p}(s), \overline{x}(s)\right)$$

5.4 Hamilton-Jacobi Equation

The initial-value problem for the Hamilton-Jacobi equation is

$$\begin{cases} u_t + H(Du) = 0 \; in \; R^n \times (0, \infty) \\ \quad u = g \qquad on \, R^n \times \{t = 0\} \end{cases}$$

Here $u : R^n \times [0, \infty) \to R$ is the unknown, $u = u(x,t)$, and $Du = D_x u = \left(u_{x_1}, \ldots, u_{x_n}\right)$. The Hamiltonian $H : R^n \to R$ and the initial function $g : R^n \to R$ are given.

Note: Two characteristic equations associated with the Hamilton-Jacobi PDE

$$u_t + H(Du, x) = 0$$

are Hamilton's ODE

$$\begin{cases} \dot{\bar{x}} = D_p H\left(\bar{p}(s), \bar{x}(s)\right) \\ \dot{\bar{p}} = -D_x H\left(\bar{p}(s), \bar{x}(s)\right) \end{cases}$$

which arise in the classical calculus of variations and in mechanics.

5.4.1 Derivation of Hamilton's ODE from a Variational Principle (Calculus of Variation)

Article: Suppose that $L : R^n \times R^n \to R$ is a given smooth function, which is called Lagrangian.

We write

$$L = L(q, x) = L(q_1, ..., q_n, x_1, ..., x_n)$$

and

$$\begin{cases} D_q L = \left(L_{q_1} ... L_{q_n} \right) \\ D_x L = \left(L_{x_1} ... L_{x_n} \right) \end{cases}$$

Where $q, x \in R^n$

For any two fix points $x, y \in R^n$ and a time $t > 0$ and we introduce the action functional

$$I\left[\bar{w}(.)\right] = \int_0^t L\left(\dot{\bar{w}}(s), \bar{w}(s)\right) ds \qquad \dots (2)$$

where the functions $\bar{w}(.) = \left(w^1(.), w^2(.), ..., w^n(.)\right)$ belonging to the admissible class

$$A = \left\{ \bar{w}(.) \in C^2\left([0, t]; R^n\right) \middle| \bar{w}(0) = y, \bar{w}(t) = x \right\}$$

Thus, a C^2 curve $\bar{w}(.)$ belongs to A if it starts at the point y at time 0 and reaches the point x at time t. According to the calculus of variations, we shall find a parametric curve $\bar{x}(.) \in A$ such that

$$I\left[\bar{x}(.)\right] = \min_{\bar{w}(.) \in A} I\left[\bar{w}(.)\right] \qquad \dots (3)$$

i.e., we are seeking a function $\bar{x}(.)$ which minimizes the functional $I[.]$ among all admissible candidates $\bar{w}(.) \in A$.

5.4.2 Theorem: Euler-Lagrange Equations

Prove that any minimizer $\bar{x}(.) \in A$ of $I[\bullet]$ solves the system of Euler-Lagrange equations

(4) $$-\frac{d}{ds}\left(D_q L\left(\dot{\bar{x}}(s), \bar{x}(s)\right)\right) + D_x L\left(\dot{\bar{x}}(s)\right) \qquad (0 \le s \le t)$$

Proof: Consider a smooth function $\bar{v}:[0,t] \to R^n$ satisfying

$$\bar{v}(0) = \bar{v}(t) = 0 \qquad \ldots (5)$$

and $\bar{v} = \left(v^1, ..., v^n\right)$

For $c \in R$, we define

$$\bar{w}(.) = \bar{x}(.) + c\bar{v}(.) \qquad \ldots (6)$$

Then, $\bar{w}(.)$ belongs to the admissible class A and $\bar{x}(.)$ being the minimizer of the action functional and so

$$I\left[\bar{x}(.)\right] \le I\left[\bar{w}(.)\right]$$

Therefore the real-valued function

$$i(c) = I\left[\bar{x}(.) + c\bar{v}(.)\right]$$

Has a minimizer at $c = 0$ and consequently

$$i'(0) = 0 \qquad \ldots (7)$$

provided $i'(0)$ exists.

Next we shall compute this derivative explicitly and we get

$$i(c) = \int_0^t L\left(\dot{\bar{x}}(s) + c\dot{\bar{v}}(s), \bar{x}(s) + c\bar{v}(s)\right) ds$$

And differentiating above equation w.r.t. c, we obtain

$$i'(c) = \int_0^t \sum_{i=1}^n L_{q_i}\left(\dot{\bar{x}} + c\dot{\bar{v}}, x + c\bar{v}\right)\dot{v}^i + L_{x_i}\left(\dot{\bar{x}} + c\dot{\bar{v}}, x + c\bar{v}\right)v^i ds$$

Set $c = 0$ and using (7), we have

$$0 = i'(0) = \int_0^t \sum_{i=1}^n L_{q_i}\left(\dot{\bar{x}}, \bar{x}\right)\dot{v}^i + L_{x_i}\left(\dot{\bar{x}}, \bar{x}\right)v^i ds \qquad \ldots (8)$$

Now we integrate (8) by parts in the first term inside the integral and using (5), we have

$$0=\sum_{i=1}^{n}\int_{0}^{t}\left[-\frac{d}{ds}\left(L_{q_i}\left(\dot{\overline{x}},\overline{x}\right)\right)+L_{x_i}\left(\dot{\overline{x}},\overline{x}\right)\right]v^i ds$$

This identity is valid for all smooth functions $\overline{v}=\left(v^1,\ldots,v^n\right)$ satisfying (5) and so

$$-\frac{d}{ds}\left(L_{q_i}\left(\dot{\overline{x}},\overline{x}\right)\right)+L_{x_i}\left(\dot{\overline{x}},\overline{x}\right)=0$$

for $0\le s\le t, i=1,\ldots,n$

Remark: We see that any minimizer $\overline{x}(.)\in A$ of $I[.]$ solves the Euler-Lagrange system of ODE. It is also possible that a curve $\overline{x}(.)\in A$ may solve the Euler-Lagrange equations without necessarily being a minimizer, in this case $\overline{x}(.)$ is a critical point of $I[.]$. So, we can conclude that every minimizer is a critical point but a critical point need not be a minimizer.

5.4.3 Hamilton's ODE:

Suppose C^2 function $\overline{x}(.)$ is a critical point of the action functional and solves the Euler-Lagrange equations. Set

(1) $$\overline{p}(s)=D_qL\left(\dot{\overline{x}}(s),\overline{x}(s)\right) \qquad \left(0\le s\le t\right)$$

where $\overline{p}(.)$ is called the generalized momentum corresponding to the position $\overline{x}(.)$ and velocity $\dot{\overline{x}}(.)$.

Now we make important hypothesis:

(2) Hypothesis: Suppose for all $x,p\in R^n$ that the equation

$$p=D_qL(q,x)$$

can be uniquely solved for *q* as a smooth function of *p* and *x*, $q \quad =\overline{q}\ge(p,x)$

Definition: The Hamiltonian H associated with the Lagrangian L is

$$H(p,x)=p.\overline{q}(p,x)-L\left(\overline{q}(p,x),x\right) \qquad \left(p,x\in R^n\right)$$

where the function $\overline{q}(.,.)$ is defined implicitly by (2).

Example: The Hamiltonian corresponding to the Lagrangian $L(q,x)=\frac{1}{2}m|q|^2-\phi(x)$ is

$$H(p,x)=\frac{1}{2m}|p|^2+\phi(x)$$

The Hamiltonian is thus the total energy and the Lagrangian is the difference between the kinetic and potential energy.

5.4.4 Theorem: Derivative of Hamilton's ODE

The functions $\bar{x}(.)$ and $\bar{p}(.)$ satisfy Hamilton's equations

$$(3) \qquad \begin{cases} \dot{\bar{x}}(s) = D_p H\left(\bar{p}(s), \bar{x}(s)\right) \\ \dot{\bar{p}}(s) = -D_x H\left(\bar{p}(s), \bar{x}(s)\right) \end{cases} \qquad (0 \le s \le t)$$

Furthermore, the mapping $s \mapsto H\left(\bar{p}(s), \bar{x}(s)\right)$ is constant.

Proof: From (1) and (2), we have

$$\dot{\bar{x}}(s) = \bar{q}\left(\bar{p}(s), \bar{x}(s)\right)$$

Let us write $\bar{q}(.) = \left(q^1(.), \ldots, q^n(.)\right)$

We compute for $i = 1, \ldots, n$

$$\frac{\partial H}{\partial x_i}(p, x) = \sum_{k=1}^{n} p_k \frac{\partial q^k}{\partial x_i}(p, x) - \frac{\partial L}{\partial q_k}(q, x)\frac{\partial q^k}{\partial x_i}(p, x) - \frac{\partial L}{\partial x_i}(q, x)$$

$$= -\frac{\partial L}{\partial x_i}(q, x) \qquad \text{(using (2))}$$

and

$$\frac{\partial H}{\partial x_i}(p, x) = q^i(p, x) + \sum_{k=1}^{n} p_k \frac{\partial q^k}{\partial p_i}(p, x) - \frac{\partial L}{\partial q_k}(q, x)\frac{\partial q^k}{\partial p_i}(p, x)$$

$$= q^i(p, x) \qquad \text{(again using (2))}$$

Thus

$$\frac{\partial H}{\partial p_i}\left(\bar{p}(s), \bar{x}(s)\right) = q^i\left(\bar{p}(s), \bar{x}(s)\right) = \dot{x}^i(s)$$

and

$$\frac{\partial H}{\partial x_i}\left(\bar{p}(s), \bar{x}(s)\right) = -\frac{\partial L}{\partial x_i}\left(\bar{q}\left(\bar{p}(s), \bar{x}(s)\right), \bar{x}(s)\right) = -\frac{\partial L}{\partial x_i}\left(\dot{\bar{x}}(s), \bar{x}(s)\right)$$

$$= -\frac{d}{ds}\left(\frac{\partial L}{\partial q_i}\left(\dot{\bar{x}}(s), \bar{x}(s)\right)\right)$$

$$= -\dot{p}^i(s)$$

Hence

$$\frac{d}{ds} H\left(\bar{p}(s), \bar{x}(s)\right) = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\dot{p}^i + \frac{\partial H}{\partial x_i}\dot{x}^i$$

$$=\sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\left(\frac{-\partial H}{\partial x_i}\right)+\frac{\partial H}{\partial x_i}\left(\frac{\partial H}{\partial p_i}\right)=0$$

which shows that the mapping $s \to H(\overline{p}(s),\overline{x}(s))$ is constant.

5.5 Legendre transform:

Assume that the Lagrangian $L: R^n \to R$ satisfies following conditions

(i) the mapping $q \mapsto L(q)$ is convex ...(1)

(ii) $\lim_{|q|\to\infty}\frac{L(q)}{|q|}=+\infty$ ... (2)

whose convexity of the mapping in equation (2) implies L is continuous.

Note: In equation (2), we simplify the Lagrangian by dropping the x-dependence in the Hamiltonian so that afterwards *H=H(p).*

Definition: The Legendre transform of L is

$$L^*(p)=\sup_{q\in R^n}\{p.q-L(q)\} \qquad (p\in R^n) \tag{3}$$

Remark: Hamiltonian H is the Legendre transform of L, and vice versa:

$$L=H^*, H=L^* \qquad \dots (4)$$

We say H and L are dual convex functions.

Theorem: Convex duality of Hamiltonian and Lagrangian

Assume L satisfies (1),(2) and define H by (3),(4)

(i)Then

the mapping $p \mapsto H(p)$ is convex

And

$$\lim_{|p|\to\infty}\frac{H(p)}{|p|}=+\infty$$

(ii)Furthermore

$$L=H^* \qquad \dots (5)$$

Proof: For each fixed q, the function $p \mapsto p.q-L(q)$ is linear, and the mapping

$$p \mapsto H(p)=L^*(p)=\sup_{q\in R^n}\{p.q-L(q)\} \text{ is convex.}$$

Indeed, if $0\le\tau\le 1,\ p.\hat{p}\in R^n$,

$$H\left(\tau p+(1-\tau)\hat{p}\right)=\sup\left\{\left(\tau p+(1-\tau)\hat{p}\right).q-L(q)\right\}$$

$$\leq \tau \sup\left\{p.q-L(q)\right\}+(1-\tau)\sup_{q}\left\{\hat{p}.q-L(q)\right\}$$

$$=\tau H(p)+(1-\tau)H(\hat{p})$$

Fix any $\lambda > 0, p \neq 0$. Then

$$H(p)=\sup_{q\in R^n}\left\{p.q-L(q)\right\}$$

$$\geq \lambda|p|-L\left(\lambda\frac{p}{|p|}\right) \qquad \left(q=\lambda\frac{p}{|p|}\right)$$

$$\geq \lambda|p|-\max_{B(0,\lambda)} L$$

Therefore, $\lim\inf_{|p|\to\infty}\dfrac{H(p)}{|p|}\geq\lambda$ for all $\lambda>0$

From (4), we have

$$H(p)+L(q)\geq p.q \qquad \forall p,q\in R^n$$

and

$$L(q)\geq\sup_{p\in R^n}\left\{p.q-H(p)\right\}=H^*(q)$$

On the other hand

$$H^*(q)=\sup_{p\in R^n}\left\{p.q-\sup\left\{p.r-L(r)\right\}\right\}$$

$$=\sup_{p\in R^n}\inf_{r\in R^n}\left\{p.(q-r)+L(r)\right\} \qquad \ldots (6)$$

since $q\mapsto L(q)$ is convex.

Let there exists $s\in R^n$ such that

$$L(r)\geq L(q)+s.(r-q) \qquad \left(r\in R^n\right)$$

Taking $p=s$ in (6)

$$H^*(q)\geq\inf_{r\in R^n}\left\{s.(q-r)+L(r)\right\}=L(q)$$

5.6 Hopf-Lax Formula

Consider the initial-value problem for the Hamilton-Jacobi equation

$$\begin{cases} u_t + H(Du) = 0 \, in \quad R^n \times (0,\infty) \\ \quad u = g \, on \quad R^n \times \{t = 0\} \end{cases} \qquad \dots (1)$$

We know that the calculus of variations problem with Lagrangian leads to Hamilton's ODE for the associated Hamilton H. Hence these ODE are also the characteristic equations of the Hamilton-Jacobi PDE, we infer there is probably a direct connection between this PDE and the calculus of variations.

Theorem: If $x \in R^n$ and $t > 0$, then the solution $u = u(x,t)$ of the minimization problem

$$u(x,t) = \inf \left\{ \int_0^t L(\dot{\bar{w}}(s))\, ds + g(y) \middle| \bar{w}(0) = y, \bar{w}(t) = x \right\} \qquad \dots (2)$$

is

$$u(x,t) = \min \left\{ tL\left(\frac{x-y}{t} \right) + g(y) \right\} \qquad \dots (3)$$

where, the infimum is taken over all C^1 functions. The expression on the right hand side of (3) called Hopf-Lax formula.

Proof: Fix any $y \in R^n$ and define

$$\bar{w}(s) = y + \frac{s}{t}(x - y) \qquad (0 \le s \le t)$$

Then $\overline{w}(0) = y$ and $\overline{w}(t) = y$

The expression (2) of u implies

$$u(x,t) \le \int_0^t L(\dot{\bar{w}}(s))\, ds + g(y) = tL\left(\frac{x-y}{t} \right) + g(y)$$

and therefore

$$u(x,t) \le \inf_{y \in R^n} \left\{ tL\left(\frac{x-y}{t} \right) + g(y) \right\}$$

If $\bar{w}(.)$ is any C^1 function satisfying $\bar{w}(t) = x$, then we have

$$L\left(\frac{1}{t} \int_0^t \dot{\bar{w}}(s)\, ds \right) \le \frac{1}{t} \int_0^t L(\dot{\bar{w}}(s))\, ds \qquad \text{(by Jensen's inequality)}$$

Thus if we write $y = w(0)$, we find

$$tL\left(\frac{x-y}{t}\right)+g(y)\leq\int_0^t L(\dot{w}(s))\,ds+g(y)$$

and consequently

$$\inf_{y\in R^n}\left\{tL\left(\frac{x-y}{t}\right)+g(y)\right\}\leq u(x,t)$$

Hence

$$u(x,t)=\inf_{y\in R^n}\left\{tL\left(\frac{x-y}{t}\right)+g(y)\right\}$$

Lemma 1: (A functional identity)

For each $x\in R^n$ and $0\leq s\leq t$, we have

$$u(x,t)=\min_{y\in R^n}\left\{(t-s)L\left(\frac{x-y}{t-s}\right)+u(y,s)\right\} \qquad \ldots (1)$$

In other words, to compute $u(.,t)$, we can calculate u at time s and then use $u(.,s)$ as the initial condition on the remaining time interval $[s,t]$.

Proof: Fix $y\in R^n, 0<s<t$ and choose $z\in R^n$ so that

$$u(y,s)=sL\left(\frac{y-z}{s}\right)+g(z) \qquad \ldots (2)$$

Now since L is convex and $\frac{x-z}{t}=\left(1-\frac{s}{t}\right)\left(\frac{x-y}{t-s}\right)+\frac{s}{t}\frac{y-z}{s}$, we have

$$L\left(\frac{x-z}{t}\right)\leq\left(1-\frac{s}{t}\right)L\left(\frac{x-y}{t-s}\right)+\frac{s}{t}L\left(\frac{y-z}{s}\right)$$

Thus

$$u(x,t)\leq tL\left(\frac{x-z}{t}\right)+g(z)\leq(t-s)L\left(\frac{x-y}{t-s}\right)+sL\left(\frac{y-z}{s}\right)+g(z)$$

$$=(t-s)L\left(\frac{x-y}{t-s}\right)+u(y,s)$$

By (2). This inequality is true for each $y\in R^n$. Therefore, since $y\mapsto u(y,s)$ is continuous, we have

$$u(x,t)\leq\min_{y\in R^n}\left\{(t-s)L\left(\frac{x-y}{t-s}\right)+u(y,s)\right\} \qquad \ldots. (3)$$

Now choose w such that

$$u(x,t) = tL\left(\frac{x-w}{t}\right) + g(w) \qquad \ldots(4)$$

and set $y := \frac{s}{t}x + \left(1-\frac{s}{t}\right)w$. Then $\frac{x-y}{t-s} = \frac{x-w}{t} = \frac{y-w}{s}$.

Consequently

$$(t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)$$

$$\leq (t-s)L\left(\frac{x-w}{t}\right) + sL\left(\frac{y-w}{s}\right) + g(w)$$

$$= tL\left(\frac{x-w}{t}\right) + g(w) = u(x,t)$$

By (4). Hence

$$\min_{y\in R^n}\left\{(t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)\right\} \leq u(x,t) \qquad \ldots(5)$$

Lemma 2: (Lipschitz continuity)

The function u is Lipschitz continuous in $R^n \times [0,\infty)$, and $u = g$ on $R^n \times \{t=0\}$.

Proof: Fix $t > 0, x, \hat{x} \in R^n$. Choose $y \in R^n$ such that

$$tL\left(\frac{x-y}{t}\right) + g(y) = u(x,t) \qquad \ldots.(6)$$

Then

$$u(\hat{x},t) - u(x,t) = \inf_z\left\{tL\left(\frac{\hat{x}-z}{t}\right) + g(z)\right\} - tL\left(\frac{x-y}{t}\right) - g(y)$$

$$\leq g(\hat{x}-x+y) - g(y) \leq Lip(g)|\hat{x}-x|$$

Hence

$$u(\hat{x},t) - u(x,t) \leq Lip(g)|\hat{x}-x|$$

and, interchanging the roles of $\hat{x}$ and x, we find

$$|u(\hat{x},t) - u(x,t)| \leq Lip(g)|x-\hat{x}| \qquad \ldots (7)$$

Now select $x \in R^n$, t>0. Choosing $y = x$ in (*), we discover

$$u(x,t) \le tL(0) + g(x) \qquad \ldots(8)$$

Furthermore,

$$u(x,t) = \min_{y \in R^n}\left\{tL\left(\frac{x-y}{t}\right) + g(y)\right\}$$

$$\ge g(x) + \min_{y \in R^n}\left\{-Lip(g)|x-y| + tL\left(\frac{x-y}{t}\right)\right\}$$

$$= g(x) - t\max_{z \in R^n}\{Lip(g)|z| - L(z)\} \qquad \left(z = \frac{x-y}{t}\right)$$

$$= g(x) - t\max_{w \in B(0,Lip(g))}\max_{z \in R^n}\{w.z - L(z)\}$$

$$= g(x) - t\max_{B(0,Lip(g))} H$$

This inequality and (8) imply

$$|u(x,t) - g(x)| \le Ct$$

For

$$\text{C:=}\max\left(|L(0)|, \max_{B(0,Lip(g))}|H|\right) \qquad \ldots (9)$$

Finally select $x \in R^n, 0 < \hat{t} < t$. Then $Lip(u(.,t)) \le Lip(g)$ by (7) above. Consequently Lemma 1 and calculations like those employed in step 2 above imply

$$|u(x,t) - u(x,\hat{t})| \le C|t - \hat{t}|$$

For the constant C defined by (9).

Theorem: Solving the Hamilton-Jacobi equation

Suppose $x \in R^n, t > 0$, and u defined by the Hopf-Lax formula

$$u(x,t) = \min_{y \in R^n}\left\{tL\left(\frac{x-y}{t}\right) + g(y)\right\}$$

is differentiable at a point $(x,t) \in R^n \times (0,\infty)$. Then

$$u_t(x,t) + H(Du(x,t)) = 0.$$

Proof: Fix $q \in R^n, h > 0$. Owing to Lemma 1,

$$u(x+hq,t+h) = \min_{y\in R^n}\left\{hL\left(\frac{x+hq-y}{h}\right)+u(y,t)\right\}$$
$$\leq hL(q)+u(x,t).$$

Hence

$$\frac{u(x+hq,t+h)-u(x,t)}{h} \leq L(q).$$

Let $h \to 0^+$, to compute

$$q.Du(x,t)+u_t(x,t) \leq L(q).$$

This inequality is valid for all $q \in R^n$, and so

$$u_t(x,t)+H(Du(x,t)) = u_t(x,t)+\max_{q\in R^n}\{q.Du(x,t)-L(q)\} \leq 0 \qquad \ldots(10)$$

The first equality holds since $H = L^*$.

Now choose z such that $u(x,t) = tL\left(\frac{x-z}{t}\right)+g(z)$. Fix h>0 and set $s = t-h, y = \frac{s}{t}x+\left(1-\frac{s}{t}\right)z$.

Then $\frac{x-z}{t} = \frac{y-z}{s}$, and thus

$$u(x,t)-u(y,s) \geq tL\left(\frac{x-z}{t}\right)+g(z)-\left[sL\left(\frac{y-z}{s}\right)+g(z)\right]$$
$$=(t-s)L\left(\frac{x-z}{t}\right)$$

That is,

$$\frac{u(x,t)-u\left(\left(1-\frac{h}{t}\right)x+\frac{h}{t}z,t-h\right)}{h} \geq L\left(\frac{x-z}{t}\right)$$

Let $h \to 0^+$ to compute

$$\frac{x-z}{t}.Du(x,t)+u_t(x,t) \geq L\left(\frac{x-z}{t}\right)$$

Consequently

$$u_t(x,t)+H(Du(x,t)) = u_t(x,t)+\max_{q\in R^n}\{q.Du(x,t)-L(q)\}$$
$$\geq u_t(x,t)+\frac{x-z}{t}.Du(x,t)-L\left(\frac{x-z}{t}\right)$$
$$\geq 0$$

This inequality and (10) complete the proof.

Lemma 3: (Semiconcavity)

Suppose there exists a constant C such that

$$g(x+z)-2g(x)+g(x-z)\le C|z|^2 \tag{11}$$

for all $x,z\in R^n$. Define u by the Hopf-Lax formula (*). Then

$$u(x+z,t)-2u(x,t)+u(x-z,t)\le C|z|^2$$

for all $x,z\in R^n, t>0$.

Remark: We say g is semiconcave provided (11) holds. It is easy to check (11) is valid if g is C^2 and $\sup_{R^n}|D^2g|<\infty$. Note that g is semiconcave if and only if the mapping $x\mapsto g(x)+\frac{C}{2}|x|^2$ is concave for some constant C.

Proof: Choose $y\in R^n$ so that $u(x,t)=tL\left(\frac{x-y}{t}\right)+g(y)$. Then putting $y+z$ and $y-z$ in the Hopf-Lax formulas for $u(x+z,t)$ and $u(x-z,t)$, we find

$$\begin{aligned}
&u(x+z,t)-2u(x,t)+u(x-z,t)\\
&\qquad\le\left[tL\left(\frac{x-y}{t}\right)+g(y+z)\right]-2\left[tL\left(\frac{x-y}{t}\right)+g(y)\right]\\
&\qquad\quad+\left[tL\left(\frac{x-y}{t}\right)+g(y-z)\right]\\
&\qquad=g(y+z)-2g(y)+g(y-z)\\
&\qquad\le C|z|^2, \qquad \text{by (11)}
\end{aligned}$$

Definition: A C^2 convex function $H:R^n\to R$ is called uniformly convex(with constant $\theta>0$) if

$$\text{(12)} \qquad \sum_{i,j=1}^{n} H_{p_ip_j}(p)\xi_i\xi_j\ge\theta|\xi|^2 \qquad \text{for all } p,\xi\in R^n$$

We now prove that even if g is not semi-concave, the uniform convexity of H forces u to become semi-concave for times t>0: it is a kind of mild regularizing effect for the Hopf-Lax solution of the initial- value problem.

Lemma 4: (Semi-concavity Again)

Suppose that H is uniformly convex (with constant θ) and u is defined by the Hopf-Lax formula. Then

$$u(x+z,t)-2u(x,t)+u(x-z,t)\le\frac{1}{\theta t}|z|^2$$

for all $x,z\in R^n, t>0$.

Proof: We note first using Taylor's formula that (12) implies

$$H\left(\frac{p_1+p_2}{2}\right)\le\frac{1}{2}H(p_1)+\frac{1}{2}H(p_2)-\frac{\theta}{8}|p_1-p_2|^2 \tag{13}$$

Next we claim that for the Lagrangian L, we have estimate

$$\frac{1}{2}L(q_1)+\frac{1}{2}L(q_2)\le L\left(\frac{q_1+q_2}{2}\right)+\frac{1}{8\theta}|q_1-q_2|^2 \tag{14}$$

For all $q_1,q_2\in R^n$. Verification is left as an exercise.

Now choose y so that $u(x,t)=tL\left(\frac{x-y}{t}\right)+g(y)$. Then using the same value of y in the Hopf-Lax formulas for $u(x+z,t)$ and $u(x-z,t)$, we calculate

$$\begin{aligned}
&u(x+z,t)-2u(x,t)+u(x-z,t)\\
&\le\left[tL\left(\frac{x+z-y}{t}\right)+g(y)\right]-2\left[tL\left(\frac{x-y}{t}\right)+g(y)\right]\\
&\quad+\left[tL\left(\frac{x+z-y}{t}\right)+g(y)\right]\\
&=2t\left[\frac{1}{2}L\left(\frac{x+z-y}{t}\right)+\frac{1}{2}L\left(\frac{x-z-y}{t}\right)-L\left(\frac{x-y}{t}\right)\right]\\
&\le 2t\frac{1}{8\theta}\left|\frac{2z}{t}\right|^2\le\frac{1}{\theta t}|z|^2,
\end{aligned}$$

The next-to-last inequality following from (14).

Theorem: Suppose $x\in R^n, t>0$, and u defined by the Hopf-Lax formula is differentiable at a point $(x,t)\in R^n\times(0,\infty)$. Then

$$u_t(x,t)+H(Du(x,t))=0$$

Proof: Fix $q \in R^n, h > 0$ and using Lemma (1), then we have

$$u(x+hq, t+h) = \min_{y \in R^n} \left\{ hL\left(\frac{x+hq-y}{h}\right) + u(y,t) \right\}$$

$$\leq hL(q) + u(x,t)$$

Hence

$$\frac{u(x+hq,t+h) - u(x,t)}{h} \leq L(q)$$

Let $h \to 0^+$, to compute

$$q.Du(x,t) + u_t(x,t) \leq L(q) \qquad \text{for all } q \in R^n$$

and therefore

$$u_t(x,t) + H(Du(x,t)) = u_t(x,t) + \max_{q \in R^n} \{ q.Du(x,t) - L(q) \} \leq 0$$

The first equality holds since $H = L*$

Now choose z such that

$$u(x,t) = tL\left(\frac{x-z}{t}\right) + g(z)$$

Fix h>0 and set

$$s = t-h, y = \frac{s}{t}x + \left(1 - \frac{s}{t}\right)z$$

Then $$\frac{x-z}{t} = \frac{y-z}{s}$$

and

$$u(x,t) - u(y,s) \geq tL\left(\frac{x-z}{t}\right) + g(z) - \left[sL\left(\frac{y-z}{s}\right) + g(z) \right]$$

$$= (t-s)L\left(\frac{x-z}{t}\right)$$

$$\Rightarrow \frac{u(x,t) - u\left(\left(1-\frac{h}{t}\right)x + \frac{h}{t}z, t-h\right)}{h} \geq L\left(\frac{x-z}{t}\right)$$

Let $h \to 0^+$ to compute

$$\frac{x-z}{t}.Du(x,t)+u_t(x,t)\ge L\left(\frac{x-z}{t}\right)$$

Consequently

$$u_t(x,t)+H(Du(x,t))=u_t(x,t)+\max_{q\in R^n}\{q.Du(x,t)-L(q)\}$$

$$\ge u_t(x,t)+\frac{x-z}{t}.Du(x,t)-L\left(\frac{x-z}{t}\right)$$

$$\ge 0$$

Hence $u_t(x,t)+H(Du(x,t))=0$

5 .7 Weak Solutions and Uniqueness

Definition: We say that a Lipschitz Continuous function $u:R^n\times[0,\infty)\to R$ is a weak solution of the initial-value problem

(15)
$$\begin{cases} u_t+H(Du)=0\, in \quad R^n\times(0,\infty) \\ u=g\ on \quad R^n\times\{t=0\} \end{cases}$$

provided

(a) $u(x,0)=g(x) \qquad (x\in R^n)$

(b) $u_t(x,t)+H(Du(x,t))=0\ for \quad a.e.(x,t)\in R^n\times(0,\infty)$

(c) $u(x+z,t)-zu(x,t)+u(x-z,t)\le c\left(1+\frac{1}{t}\right)|z|^2$

for some constant $c\ge 0$ and all $x,z\in R^n, t>0$.

Theorem: Uniqueness of Weak Solution

Assume H is C^2 and satisfies $\begin{cases} H\ is \quad convex \quad and \\ \lim\limits_{|p|\to\infty}\frac{H(p)}{|p|}=+\infty \end{cases}$ and $g:R^n\to R$ is Lipschitz continuous. Then there exists at most one weak solution of the initial-value problem (15).

Proof: Suppose that u and $\tilde{u}$ are two weak solutions of (15) and write $w:=u-\tilde{u}$.

Observe now at any point (y,s) where both u and $\tilde{u}$ are differentiable and solve our PDE, we have

$$w_t(y,s)=u_t(y,s)-\tilde{u}_t(y,s)$$

$$= -H\left(Du(y,s)\right) + H\left(D\tilde{u}(y,s)\right)$$

$$= -\int_0^1 \frac{d}{dr} H\left(rDu(y,s) + (1-r)D\tilde{u}(y,s)\right) dr$$

$$= -\int_0^1 DH\left(rDu(y,s) + (1-r)D\tilde{u}(y,s)\right) dr.\left(Du(y,s) - D\tilde{u}(y,s)\right)$$

$$=: -b(y,s).Dw(y,s)$$

Consequently

$$w_t + b.Dw = 0 \quad \text{a.e.} \qquad \text{… (16)}$$

Write $v := \phi(w) \geq 0$, where $\phi : R \to [0,\infty)$ is a smooth function to be selected later. We multiply(16) by $\phi'(w)$ to discover

$$v_t + b.Dv = 0 \qquad \text{a.e.} \qquad \text{…(17)}$$

Now choose $\varepsilon > 0$ and define $u^\varepsilon := \eta_\varepsilon * u, \tilde{u}^\varepsilon := \eta_\varepsilon * \tilde{u}$, where η_ε is the standard mollifier in the x and t variables. Then we have

$$\left|Du^\varepsilon\right| \leq Lip(u), \left|D\tilde{u}^\varepsilon\right| \leq Lip(\tilde{u}), \qquad \text{… (18)}$$

and

$$Du^\varepsilon \to Du, D\tilde{u}^\varepsilon \to D\tilde{u} \quad \text{a.e., as } \varepsilon \to 0 \qquad \text{…(19)}$$

Furthermore inequality(c) in the definition of weak solution implies

$$D^2u^\varepsilon, D^2\tilde{u}^\varepsilon \leq C\left(1 + \frac{1}{s}\right)I$$

For an appropriate constant C and all $\varepsilon > 0$, $y \in R^n$, $s > 2\varepsilon$. Verification is left as an exercise.

Write

$$b_\varepsilon(y,s) := \int_0^1 DH\left(rDu^\varepsilon(y,s) + (1-r)D\tilde{u}^\varepsilon(y,s)\right) dr \qquad \text{… (20)}$$

Then (17) becomes

$$v_t + b_\varepsilon.Dv = (b_\varepsilon - b).Dv \qquad \text{a.e.}$$

Hence

$$v_t + div(vb_\varepsilon) = (divb_\varepsilon)v + (b_\varepsilon - b).Dv \quad \text{a.e.} \qquad \text{…(21)}$$

Now

$$divb_\varepsilon = \int_0^1 \sum_{k,l=1}^{n} H_{p_k p_l}\left(rDu^\varepsilon + (1-r)D\tilde{u}^\varepsilon\right)\left(ru^\varepsilon_{x_l x_k} + (1-r)\tilde{u}^\varepsilon_{x_l x_k}\right)dr$$

$$\leq C\left(1+\frac{1}{s}\right) \qquad \ldots(22)$$

For some constant C, in view of (17) and (19). Here we note that H convex implies $D^2H \geq 0$.

Fix $x_0 \in R^n, t_0 > 0$, and set

$$R := \max\left\{\left|DH(p)\right| \,\middle|\, |p| \leq \max\left(Lip(\tilde{u})\right)\right\} \qquad \ldots(23)$$

Define also the cone

$$C := \left\{(x,t) \,\middle|\, 0 \leq t \leq t_0, |x-x_0| \leq R(t_0-t)\right\}$$

Next write

$$e(t) = \int_{B(x_0,R(t_0-t))} v(x,t)\,dx$$

and compute for a.e. t>0:

$$\dot{e}(t) = \int_{B(x_0,R(t_0-t))} v_t\,dx - R\int_{\partial B(x_0,R(t_0-t))} v\,dS$$

$$= \int_{B(x_0,R(t_0-t))} -div(vb_\varepsilon) + (divb_\varepsilon)v + (b_\varepsilon - b).Dv\,dx$$

$$-R\int_{\partial B(x_0,R(t_0-t))} v\,dS \qquad \text{by (21)}$$

$$= -\int_{\partial B(x_0,R(t_0-t))} v(b_\varepsilon.v + R)\,dS$$

$$+\int_{B(x_0,R(t_0-t))} (divb_\varepsilon)v + (b_\varepsilon - b).Dv\,dx$$

$$\leq \int_{B(x_0,R(t_0-t))} (divb_\varepsilon)v + (b_\varepsilon - b).Dv\,dx \qquad \text{by (17), (20)}$$

$$\leq C\left(1+\frac{1}{t}\right)e(t) + \int_{B(x_0,R(t_0-t))} (b_\varepsilon - b).Dv\,dx$$

by (22). The last term on the right hand side goes to zero as $\varepsilon \to 0$, for a.e. $t_0 > 0$, according to (17), (18) and the Dominated Convergence Theorem.

Thus

$$\dot{e}(t) \le C\left(1+\frac{1}{t}\right)e(t) \quad \text{for a.e. } 0 < t < t_0 \tag{24}$$

Fix $0 < \varepsilon < r < t$ and choose the function $\phi(z)$ to equal zero if

$$|z| \le \varepsilon\left[Lip(u) + Lip(\tilde{u})\right]$$

and to be positive otherwise. Since $u = \tilde{u}$ on $R^n \times \{t = 0\}$,

$$v = \phi(w) = \phi(u - \tilde{u}) = 0 \quad \text{at } \{t = \varepsilon\}$$

Thus $e(\varepsilon) = 0$. Consequently Gronwall's inequality and (24) imply

$$e(r) \le e(\varepsilon) e^{\int_\varepsilon^r C\left(1+\frac{1}{s}\right)dS} = 0$$

Hence

$$|u - \tilde{u}| \le \varepsilon\left[Lip(u) + Lip(\tilde{u})\right] \quad \text{on } B\left(x_0, R\left(t_0 - r\right)\right)$$

This inequality is valid for all $\varepsilon > 0$, and so $u \equiv \tilde{u}$ in $B\left(x_0, R\left(t_0 - r\right)\right)$. Therefore, in particular, $u(x_0, t_0) = \tilde{u}(x_0, t_0)$.